CONTINUUM SEMIGROUPS WITH MIDUNIT

by

HUNG-TZAW HU

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF THE
UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1974

to

My Dear Parents

with Love

ACKNOWLEDGEMENTS

The author wishes to express his indebtedness and sincere gratitude to Professor Alexander D. Wallace, chairman of the supervisory committee prior to his retirement in June 1973, for sharing his wisdom, great experience and philosophy as well as his professional guidance throughout the primary preparation of this work. He would like to express his appreciation to Professor Kermit N. Sigmon for agreeing to chair his supervisory committee since September 1973 after his return from Germany. Words can hardly express the depth and sincerity of the author's thanks to Professor Sigmon not only for his kind guidance and interest in developing the writing ability of the author but also for his critical comments, valuable suggestion, and new ideas which made this work to be done in a possible frank and perfect form. The author also takes this opportunity to extend his sincere gratitude to Professors P. Bacon and David Stadtlander for their introduction of the theory of cohomology and the theory of compact semigroups, which are essential for this work. He also wishes to thank Professors Thomas Bowman, J. A. Draper, David Stadtlander, E. E. Shult, and Mark Yang for serving on his supervisory commmittee. A more profound acknowledgement is due to Professor Thomas Bowman for his useful suggestions and encouragement

during the preparation of this work. Finally he extends his thanks to Professor A. R. Bednarek, Chairman of the Department of Mathematics, for his financial support during the past years.

TABLE OF CONTENTS

Page

ACKNOWLEDGEMENTS....iii

ABSTRACT....vi

INTRODUCTION....1

CHAPTERS

I PRELIMINARIES....5

II BASIC PROPERTIES OF MIDUNITS....10

III THE $\mathcal{J}$-CLASS OF MIDUNITS IN A COMPACT SEMIGROUP..15

IV FINITE SEMIGROUPS WITH MIDUNIT....20

V THE CODIMENSION OF THE $\mathcal{J}$-CLASS OF MIDUNITS IN A CONTINUUM SEMIGROUP....28

VI THE COMPLEMENT OF THE $\mathcal{J}$-CLASS OF MIDUNITS AND MINIMAL CONTINUUM SEMIGROUPS WITH MIDUNIT....34

VII A METHOD OF CONSTRUCTING SEMIGROUPS WITH MIDUNIT....39

VIII SEMIGROUPS ON D-SPACES....44

IX EXAMPLES OF SEMIGROUPS ADMITTING NO PROPER MIDUNIT....54

X SEMIGROUPS WITH MIDUNIT ON THE TWO-CELL....64

XI MARGINALITY OF THE MIDUNITS IN A CONTINUUM SEMIGROUP....71

XII ACYCLICITY OF CONTINUUM SEMIGROUPS WITH MIDUNIT....74

REFERENCES....81

BIOGRAPHICAL SKETCH....83

Abstract of Dissertation Presented to the
Graduate Council of the University of Florida in
Partial Fulfillment of the Requirements for the
Degree of Doctor of Philosophy

CONTINUUM SEMIGROUPS WITH MIDUNIT

By

Hung-tzaw Hu

August, 1974

Chairman: K. N. Sigmon
Major Department: Department of Mathematics

A continuum semigroup with midunit v is defined as a compact connected semigroup satisfying avb = ab for all a and b. It is the purpose of this dissertation to investigate continuum semigroups with midunit. If S is a compact semigroup with midunit satisfying $S^2 = S$ and J is the $\mathcal{J}$-class of the midunits, then J is the unique maximal $\mathcal{J}$-class in S and is a completely simple subsemigroup whose idempotents are the set of all midunits and form a usual rectangular band. There exists an example of a semigroup with five elements which satisfies the condition $S^2 = S$ and has a unique maximal $\mathcal{J}$-class, but S does not have a midunit. The cohomology used is that of Alexander-Wallace-Spanier with coefficient group arbitrary unless otherwise specified. It is shown that if S is a continuum semigroup of finite codimension, then $cd(H_a) < cd(S)$ for all a in S such that $H_a \neq K$. It follows that the $\mathcal{J}$-class of the midunits in a continuum semigroup of codimension one satisfying $S^2 = S \neq K$ is totally disconnected. There is a method of constructing a new semigroup with midunit from a given known semigroup, which is used to prove that if S is a compact semigroup, v is a $\mathcal{J}$-maximal idempotent,

J is the $\mathcal{J}$-class of v and $vJv \subset J$, then J is homeomorphic to a cartesian product of a rectangular band with H_v, the maximal subgroup which contains v. A midunit is called proper if it is neither a left nor a right identity. A certain class of spaces is given which, due to their topological structure, do not admit a semigroup with proper midunit under the condition $S^2 = S \neq K$. Let S be a semigroup on the two-cell with midunit and zero satisfying $S^2 = S$, M be the set of all midunits, and H_v be the maximal subgroup containing the midunit v. If either $H^1(H_v) \neq 0$ or $H^1(M) \neq 0$, then all midunits lie on the boundary of the two-cell and are either left or right identities for S. If S is a continuum semigroup with midunit v satifying $S^2 = S \neq K$ and H_v and $[(Sv \cup vS)\setminus vSv]^*$ are disjoint then each point of H_v is marginal in S. An example is given of a continuum semigroup with zero and midunit satisfying $S^2 = S$ which is not acyclic. It is shown, however, that if S is a continuously factorable continuum semigroup with zero and midunit, then S is acyclic. By a continuously factorable semigroup is meant a semigroup $(S;m)$ for which there exists a map η from S onto $S \times S$ such that $m \circ \eta = I_S$.

INTRODUCTION

The purpose of this chapter is to introduce the motivation for this work and to give a summary and key results obtained in this investigation as well as some unsolved problems. The author was attracted to the idea of midunit when he presented a report on an example concerning midunit from the dissertation of McCharen [11] in the seminar of Professor A. D. Wallace at the University of Florida two years ago. The underlying space of this example consists of countable many tangent circles converging to a point. He proved that if it admits the structure of a semigroup S satisfying $S^2 = S$, then S = K. His proof used a method which revealed the wonder and attraction of the notion of midunit both in the algebraic and topological sense. The idea of midunit was first introduced by Yamada in his paper "A note on middle unitary semigroups" nineteen years ago. Recently Ault has published a paper in this area in which she found that every (regular) semigroup can be embedded in a single (regular) semigroup with midunits which are neither left nor right identities [1].

After giving topological and algebraic preliminaries in Chapter I, we introduce the main notion of this work, which is midunit, and discuss some basic properties of it in Chapter II. Among these basic properties of midunits we describe the structure of the set of idempotent midunits and the union of all maximal groups which contain a midunit as identity, as well as the interrelation between them. Chapter III is mainly an investigation of the structure and interrelation of J and M in a compact semigroup S with midunit satisfying $S^2 = S$, where J is the $\mathcal{J}$-class of the midunits and M is the set of all midunits. In this direction we prove that J is the unique maximal $\mathcal{J}$-class in S, which is a completely simple semigroup whose idempotents are midunits and form a rectangular band. Professor Sigmon raised the question of whether a compact semigroup S satisfying $S^2 = S$, with a unique maximal $\mathcal{J}$-class forming a completely simple semigroup must contain a midunit. The answer to this question is given in Chapter IV. More precisely, we prove that if S contains at most four elements and satisfies $S^2 = S$, then any idempotent v satisfying SvS = S must be a midunit. On the other hand we give an example of a semigroup with five elements which satisfies the condition $S^2 = S$ and has a unique maximal $\mathcal{J}$-class which is a completely simple subsemigroup, but S does not have a midunit. As a continuation of Chapter III, we prove in Chapter V that the $\mathcal{J}$-class J of the midunits in a continuum semigroup of codimension one satisfying $S^2 = S \neq K$ is totally disconnected. In the process of proving this, we present a rather short proof of an interesting result; namely, that if S is a continuum semigroup with finite codimension, then $cd(H_a) < cd(S)$ for all a in S such that $H_a \neq K$.

The structure of the complement of the $\mathcal{J}$-class of midunits in a continuum semigroup satisfying $S^2 = S \neq K$ will be studied in Chapter VI. Also as a generalization of the notion of irreducible clan, we introduce and investigate the structure of minimal continuum semigroups with midunits in the latter part of Chapter VI. In Chapter VII we study in detail a method of constructing new semigroups with midunits from given known semigroups. The importance of this method is that it could transfer problems in general semigroups into problems of semigroups with midunits and vice-versa. By using this method and results of preceding chapters it is not difficult to prove that if S is a compact semigroup, v is a $\mathcal{J}$-maximal idempotent, J is the $\mathcal{J}$-class of v, and $vJv \subset J$, then J is homeomorphic to a cartesian product of a rectangular band with H_v, the maximal subgroup which contains v. Furthermore, if $\mathrm{cd}S = 1$ and $S \neq K$, then J is totally disconnected.

In Chapter VIII and Chapter IX we discuss those semigroups which, due to their topological structure, do not admit a proper midunit under the condition $S^2 = S \neq K$, In this class of spaces there are D-spaces, a circle with a closed interval issuing from a point on the circle, etc. Chapter X is devoted to exploring those semigroups S on the two-cell with midunit and zero satisfying $S^2 = S$. The cohomology theory used is that of Alexander-Wallace-Spanier with arbitrary coefficient group unless otherwise specified. Let S be a semigroup on the two-cell with midunit and zero satisfying $S^2 = S$, M be the set of all midunits, and H_v be the maximal group containing the midunit v. If either $H^1(H_v) \neq 0$ or $H^1(M) \neq 0$, then every midunit must be either left or right identity for S.

The last two chapters deal with cohomological results in continuum semigroups with midunit satisfying $S^2 = S \neq K$ which are analogous to those holding for continuum semigroups with identity. The problem of whether a midunit in a continuum semigroup satisfying $S^2 = S \neq K$ must be marginal remains open. We prove however, in Chapter XI that if $H_v \cap [(Sv \cup vS)\backslash vSv]^* = \square$, where H_v is the maximal group containing v, then each point of H_v in marginal in S. In Chapter XI we are concerned with the acyclicity of a continuum semigroup with midunit. An example is given of a continuum semigroup with zero and midunit satisfying $S^2 = S$ which is not acyclic. We introduce, however, the notion of a continuously factorable semigroup and prove that if S is a continuously factorable continuum semigroup with zero and midunit, then S is acyclic. We end this introduction with an unsolved problem: Must a continuously factorable continuum semigroup with zero be acyclic?

CHAPTER I

PRELIMINARIES

Topological Preliminaries. Throughout this work a *space* will always be a Hausdorff topological space. If A and B are subsets of a space X, then the complement of B in A will be denoted by $A\setminus B$. The empty set is denoted by $\square$. If A is a subset of the space X, then the closure of A in X will be denoted by A^*, the interior by A^o, and the boundary $(A^*\setminus A^o)$ by $F(A)$. A is said to be *nowhere dense* if $(A^*)^o = \square$. A *continuum* is a compact connected Hausdorff space. The terms *map* and continuous function are synonomous. If a map is one-to-one, onto and open it is a *homeomorphism*. The term *iff* will mean "if and only if".

Let X and Y be spaces. A *relation* from X to Y is a subset of the cartesian product $X \times Y$. A *relation* on X is a subset of $X \times X$. If Γ is a closed subset of $X \times Y$, we call Γ a *closed relation* from X to Y. If Γ is a relation on X, then $\Gamma^{-1} = \{(y,x) \mid (x,y) \in \Gamma\}$. The *diagonal* of $X \times X$ will be denoted by Δ_X; that is, $\Delta_X = \{(x,x) \mid x \in X\}$. The relation Γ on X is *reflexive* if $\Delta_X \subset \Gamma$, *symmetric* if $\Gamma = \Gamma^{-1}$, *anti-symmetric* if $\Gamma \cap \Gamma^{-1} \subset \Delta_X$, and transitive if $(x,y) \in \Gamma$ and $(y,z) \in \Gamma$ imply $(x,z) \in \Gamma$. A *quasi-order* is a reflexive transitive relation. A *partial order* is an anti-symmetric quasi-order. An *equivalence relation* is a symmetric quasi-order. For an equivalence relation Γ on a space X, X/Γ denotes the set of equivalence classes of Γ. If X is compact and Γ is closed, X/Γ endowed with the quotient topology is a space. If A is a closed subset of the space, we may define Γ on X to be the set of all ordered pairs (x,y) in $X \times X$ such that either $x = y$ or x and y are elements of A. Then the space X/Γ is written X/A.

If Γ is a relation on X and b is a point of X, then $L(b) = \{b\} \cup \{x \mid (x,b) \in \Gamma\}$, $M(b) = \{b\} \cup \{y \mid (b,y) \in \Gamma\}$, and $L_b = L(b) \cap M(b)$. If A is a subset of X, then $L(A)$ denotes the union of all sets $L(a)$ for which $a \in A$. It is easily verified that if X is compact and Γ is closed, $L(A)$ is closed for each closed subset A of X.

The cohomology theory used is that Alexander-Wallace-Spanier with arbitrary coefficient group unless otherwise specified. Throughout this work we shall use reduced groups in dimension 0. On occassion we shall require certain standard results concerning the cohomology of compact spaces as follows [13].

1-1 Mayer-Vietoris Sequence Theorem. If for each quadruple $(X;A,B;C)$, where X is fully normal, A and B are closed subsets of X with $X = A \cup B$, and $C \subset A \cap B$, we set $\Delta = j^* k^{*-1} \delta$ where $\delta\colon H^{p-1}(A \cap B,C) \to H^p(A,A \cap B)$, $k\colon (A,A \cap B) \subset (X,B)$, and $j\colon (X,C) \subset (X,B)$, then the following sequence

$$H^p(A \cap B,C) \overset{\Delta}{\to} H^p(X,C) \xrightarrow{t^* \times t'^*} H^p(A,C) \times H^p(B,C) \xrightarrow{i^* - i'^*}$$

$H^p(A \cap B,C) \to$ is exact, where i, i′, t, and t′ are inclusion maps.

In particular, this holds when X is compact, since a compact space is fully normal.

Let X be a space and $h \in H^p(X)$. If $A \subset X$, then $h|A$ denotes the image of h under the natural homomorphism $H^p(X) \xrightarrow{i^*} H^p(A)$.

1-2 Reduction Theorem. Suppose A is closed in the fully normal space X and $h \in H^p(X)$. If $h|A = 0$, then there is an open set M containing A such that $h|M^* = 0$.

Let X be a space and A be a subset of X. Let $i\colon A \to X$ be the inclusion map and $h \in H^p(A)$. Then h is *extendible* to $H^p(X)$ if $h \in \operatorname{Im}(i^*)$. A closed subset R of X is a *roof* for h if h is not extendible to $H^p(R \cup A)$ while R is extendible to $H^p(R' \cup A)$ for any proper closed subset R' of R.

1-3 Roof Theorem. Suppose X is a compact space, A is a closed subset of X and $h \in H^p(A)$.

(1) If h is not extendible to $H^p(X)$, then h has a roof.

(2) If R is a roof for h and $h_o = h|R \cap A$, then $h_o \neq 0$, R is a roof for h_o, $R = (R\backslash A)^*$, and $R\backslash A$ is connected.

Two maps $f,g\colon X \to Y$ are *weakly homotopic* if there is a continuum T, a map $F\colon X \times T \to Y$ and a pair of elements t_o and t_1 of T such that, for all x of X, $f(x) = F(x,t_o)$ and $g(x) = F(x,t_1)$ [14].

1-4 Homotopy Theorem. If X is a compact space and if $f,g\colon X \to Y$ are weakly homotopic maps, then

$$f^* = g^*\colon H^p(Y) \to H^p(X) \quad \text{for each } p.$$

Topological Algebraic Preliminaries. Throughout this work a *topological semigroup* $(S;m)$ will be a space S endowed with a continuous, association multiplication m. If no confusion seems likely, we simply let xy denote $m(x,y)$ and $AB = \{xy \mid x \in A \text{ and } y \in B\}$, where A and B are subsets of S. An idempotent is an idempotent e such that $e^2 = e$. The set of idempotents will be denoted by E. Each idempotent e belongs to a unique maximal subgroup of S, denoted by H_e, and distinct idempotents have disjoint maximal subgroups. A nonempty set A is a left (right, two-sided) ideal of S provided $SA \subset A$ ($AS \subset A$, $AS \cup SA \subset A$). It is well known that a compact semigroup S has a unique minimal two-sided ideal, denoted by K, which is known to be a retract and is a union of maximal groups [17]. Let S be a semigroup. An element v is a left (right) identity if $va = a$ ($av = a$) for all a in S; v is an identity if it is both a left and right identity of S [2]. A clan is a continuum semigroup with identity. A clan S with identity v is called irreducible if it does not contain a proper subcontinuum semigroup T containing v with $T \cap K \neq \square$. For more information about irreducible clans the reader is referred to [6] and [15].

The following is a result due to Wallace [16] which will be of importance in the sequel.

1-5 Swelling Lemma. Let S be a compact semigroup and A a closed subset of S. If $A \subset Ax$ for some $x \in S$, then $A = Ax$.

If S is compact, then maximal elements exist and each element of S is below a maximal element. Also if S is compact and a is $\mathcal{J}$-maximal, then a is $\mathcal{L}$-maximal as well as $\mathcal{R}$-maximal [11]. The following theorem, due to McCharen [11], concerning maximal elements in a compact semigroup will be important in the sequel.

1-6 Theorem. If S is a compact semigroup satisfying $S^2 = S$ and b is a $\mathcal{J}$-maximal element of S, then $b \in ESE$.

A point p in a continuum X is a *weak cut point between a and b* if a and b are points in X different from p, and any subcontinuum of X containing a and b also contains p. The point p is simply a *weak cut point* if there exists a and b such that p is a weak cut point between a and b [21].

1-7 Theorem. Let S be a continuum semigroup satisfying $S^2 = S$. If e is a $\mathcal{J}$-maximal idempotent of S which is a weak cut point of S, then $S = K$.

A point p in a continuum X is a *local separating point* of X if p separates the component containing p of some closed neighborhood of p [21].

1-8 Theorem. Let S be a continuum semigroup with $S^2 = S$, a an $\mathcal{L}$-maximal element of S, and $a \in SE$. If a is a local separating point of S, then $a \in K$.

It is true that if S is compact semigroup and a is $\mathcal{J}$-maximal then a is $\mathcal{L}$-maximal. It follows from Theorem 1-6, that if S is a continuum semigroup satisfying $S^2 = S \neq K$, and b is a $\mathcal{J}$-maximal element of S, then b is not a local separating point of S.

Also under some conditions on S, a $\mathcal{J}$-maximal idempotent cannot have a two-dimensional Euclidean neighborhood as follows.

1-9 Theorem. Let S be a continuum semigroup satisfying $S = ESE$, and suppose e is a $\mathcal{J}$-maximal idempotent element of S. If S is a subset of the complex plane $\mathbb{C}$, $S \neq K$, then e lies on the boundary of S in $\mathbb{C}$.

CHAPTER II

BASIC PROPERTIES OF MIDUNITS

It is our purpose in this chapter to introduce the main notion of this work, which is midunit, and discuss some basic properties of it. In particular, we study the structure of the set of all idempotent midunits and the union of all maximal subgroups which contain an idempotent midunit as an identity. A point v of a semigroup S is called a midunit if avb = ab for any pair of elements a and b of S. It is evident that a one-sided identity is a midunit, while a midunit may not even be an idempotent. A midunit v of a semigroup S is proper if it is neither a left identity nor a right identity of S. We begin with some examples of semigroups in which a midunit could exist without being a left identity, right identity, or even an idempotent.

2-1 Example. Let S be topologically the closed unit interval [0,1]. We define two multiplications on S as follows:

(1) Define ab = 0 for any pair of elements a and b of S. Then every point of S is a midunit and 0 is the only idempotent midunit of S.

(2) Define ab = minimum $\{\frac{1}{2},a,b\}$ for any pair of elements a and b of S. Then every point of $[\frac{1}{2},1]$ is a midunit of S but only $\frac{1}{2}$ is an idempotent midunit.

Besides showing the existence of proper midunits these examples also show that a midunit can occur more pathologically than a one-sided identity, not only in an algebraic but also in a topological sense.

The following is a lemma concerning idempotent midunits which is also known to Janet E. Ault [1].

2-2 Lemma. Let S be a semigroup with midunit. If a and b are midunits of S, then ab is an idempotent midunit.

2-3 Lemma. Let S be a semigroup with midunit b. If for some point d, bd is a midunit of S, then d is a midunit and bd is an idempotent.

Proof. Let x and y be a pair of elements of S. Then $xdy = xb(dy) = x(bd)y = xy$, i.e., d is a midunit of S. By Lemma 2-2 we know that bd is an idempotent.

Let S and $\overline{S}$ be two semigroups. A mapping ψ of S into $\overline{S}$ is called a homomorphism if $\psi(ab) = \psi(a)\psi(b)$ for all a, b in S. A one-to-one homomorphism ψ of S onto $\overline{S}$ is called an isomorphism of S onto $\overline{S}$. If v is a midunit of S and ψ is a homomorphism from S onto $\overline{S}$, then $\psi(v)$ is a midunit of $\overline{S}$. A semigroup is called a band if all points of S are idempotent. A semigroup S is called a rectangular band if S is isomorphic to $L \times R$, where L is a semigroup with left zero multiplication, R is a semigroup with right zero multiplication and $L \times R$ is a semigroup with coordinatewise multiplication.

2-4 Proposition. Let S be a semigroup with midunit and M be the set of all idempotent midunits of S. If v is an element of M then

(1) Mv is a subsemigroup with left zero multiplication and vM is a subsemigroup with right zero multiplication.

(2) Let $Mv \times vM$ have coordinatewise multiplication, i.e., $(u,w)(\bar{u},\bar{w}) = (u\bar{u},w\bar{w})$, and ψ: $Mv \times vM \to M$ be defined by $\psi((u,w)) = uw$. Then ψ is an isomorphism. Hence, M is a rectangular band.

Proof. (1) From Lemma 2-2 we know that M is not empty. Let u and w be two elements of Mv. Then $uw = (uv)wv = (uv)v = uv^2 = u$, i.e., Mv is a subsemigroup with left zero multiplication. Similarly we could prove that vM is a subsemigroup with right zero multiplication.

(2) To prove that ψ is a homomorphism, we let (u,w) and $(\bar{u},\bar{w})$ be two elements of $Mv \times vM$. Then

$$\psi((u,w))\psi((\bar{u},\bar{w})) = (uw)(\overline{uw}) = (uw)\bar{w} = u\bar{w}, \text{ and}$$

$$\psi((u,w)(\bar{u},\bar{w})) = \psi((u\bar{u},w\bar{w})) = \psi((u,\bar{w})) = u\bar{w}.$$

Then ψ is a homomorphism. To prove that ψ is a one-to-one mapping, we let (u,w), $(\bar{u},\bar{w})$ be any two elements of $Mv \times vM$ such that $\psi((u,w)) = \psi((\bar{u},\bar{w}))$, i.e., $uw = \overline{uw}$. Then $uwv = uv = u$ and $\overline{uwv} = \overline{uv} = \bar{u}$, but $uwv = \overline{uwv}$ hence $u = \bar{u}$. Similarly we can prove that $w = \bar{w}$. It is easy to see that ψ is an onto mapping, since any idempotent u can be written as $u = u^2 = uvvu = \psi((uv,vu))$. The proof is complete.

Recall that if S is a semigroup and b an element of S, the smallest left (right, two-sided) ideal containing b is denoted by $L(b)$, $(R(b), J(b))$, and the $\mathcal{H}$-class which contains b is denoted by H_b. If b is an idempotent then H_b is the maximal subgroup of S which contains b.

2-5 Lemma. Let S be a semigroup with midunits. If u and v are two idempotent midunits of S, then $H_u H_v \subseteq H_{uv}$.

Proof. Let $x \in H_u$ and $y \in H_v$ we want to prove that $xy \in H_{uv}$. By the definition of $\mathcal{H}$-class, we have

$$xS = uS,\ Sx = Su,\ yS = vS \text{ and } Sy = Sv.$$

Then $L(xy) = xy \cup Sxy = uxy \cup Sxy = Sxy = Suy = Sy = Sv = Suv$

$= L(uv)$, and similarly $R(xy) = R(uv)$.

This implies that $xy \in H_{uv}$.

If S is a semigroup without zero, then S is simple if S does not properly contain an ideal. Further, S is completely simple if S is simple and contains a primitive idempotent. An idempotent e is called primitive if $ef = fe = f$, $f^2 = f$ implies $e = f$ [2].

2-6 Proposition. Let S be a semigroup with midunits and M be the set of all idempotent midunits of S. If $N = \bigcup_{v \in M} H_v$, then N is a completely simple subsemigroup, in which M is a subsemigroup. More precisely, let v be a midunit and consider $Mv \times H_v \times vM$ with coordinatewise multiplication. If $\psi: Mv \times H_v \times vM \to N$ is defined by $\psi(u,x,w) = uxw$, then ψ is an isomorphism under which $\psi[Mv \times \{v\} \times vM] = M$.

Proof. From Lemma 2-5, we know that N is a subsemigroup. If P is an ideal of N, then it is evident that $P \cap M \neq \sqcap$. Let $v \in P \cap M$, and z be an element in N. Then $z \in H_w$ for some $w \in M$, i.e., $z = wz$. Since v is a midunit we have $wvz = wz = z$, thus $z \in P$. This proves that $P = N$. In order to prove that the midunit v is primitive, we let $u \in M$ such that $uv = vu = u$. Then $v = (vu)v = (uv)v = uv = u$, which implies that v is primitive. Hence N is a completely simple subsemigroup. The proof of ψ is an isomorphism and $\psi[Mv \times \{v\} \times vM] = M$ are essentially the same as what we did in Proposition 2-4.

CHAPTER III

THE $\mathcal{J}$-CLASS OF MIDUNITS IN A COMPACT SEMIGROUP

In this chapter we investigate the structure and interrelation of J and M in a compact semigroup S with midunit satisfying $S^2 = S$, where J is the $\mathcal{J}$-class of the midunits and M is the set of all midunits. We note here that under the condition $S^2 = S$ all midunits belong to the same $\mathcal{J}$-class as described in Proposition 3-1. From Theorem 1-6 we have that if S is a compact semigroup satisfying $S^2 = S$ and z is a $\mathcal{J}$-maximal element of S then there exists a pair of idempotents e and f such that z = ezf.

3-1 Proposition. Let S be a compact semigroup with midunit satisfying $S^2 = S$. Then

(1) There is only one maximal $\mathcal{J}$-class in S and this $\mathcal{J}$-class contains all midunits.

(2) All midunits are idempotent.

Proof. Let z be a midunit in S. Then

$$J(z) = z \cup Sz \cup zS \cup SzS \supset SzS = S^2 = S; \quad \text{thus } J(z) = S.$$

This implies that all midunits are in the same $\mathcal{J}$-class, which is the only maximal $\mathcal{J}$-class in S. Let e and f be two idempotents such that z = ezf. Then z = ez so that by Lemma 2-3 we know that z is an idempotent.

3-2 Remark. Without the algebraic condition $S^2 = S$, none of the results in Proposition 3-1 remain true. Recall in Example 2-1 part (2) that $S^2 = [0,\frac{1}{2}] \neq S$ and every point of $[\frac{1}{2},1]$ is a midunit of S while only $\frac{1}{2}$ is an idempotent midunit. On the other hand, $J(\frac{1}{2}) = [0,\frac{1}{2}]$ and $J(1) = [0,\frac{1}{2}] \cup \{1\}$. Hence midunits 1 and $\frac{1}{2}$ are not even in the same $\mathcal{J}$-class.

The following lemma is due to Koch and Wallace. For its importance to this work and convenience to the reader, we repeat its proof [20].

3-3 Lemma. Let S be a compact semigroup. If x, y are two points in S and $L(x) \subset L(y) \subset J(x)$, then $L(x) = L(y)$.

Proof. To avoid the triviality of its proof we may assume that $y \neq x$. $y \in J(x)$ implies that $y = xt_1$, $y = t_2x$ or $y = t_3xt_4$ for some elements t_1, t_2, t_3 and t_4 of S. In case $y = t_2x$, then $L(y) \subset L(x)$, by assumption $L(x) \subset L(y)$, we have the equality $L(x) = L(y)$. In case $y = t_3xt_4$, then $L(y) = t_3xt_4 \cup St_3xt_4$; thus $L(x) \subset L(y) \subset L(x)\ t_4$ so that $L(x) = L(y)$ by the Swelling Lemma 1-5. If $y = xt_1$, the proof is similar.

The following hypothesis is held fixed throughout the remainder of this chapter:

S is a compact semigroup with midunit satisfying $S^2 = S$.

We let J be the $\mathcal{J}$-class of the midunits, M be the set of all midunits, and H_v be the $\mathcal{H}$-class of midunit v.

3-4 Proposition. $x \in J$ iff $SxS = S$.

Proof. (⇒) If x is an element of J, then by Theorem 1-6 there exist a pair of idempotents e and f such that $x = exf$.

Since $x \cup xS \cup Sx = exf \cup exS \cup Sxf \subset SxS$, we have $J(x) = x \cup xS \cup Sx \cup SxS = SxS$. Also since x is in the $\mathcal{J}$-class of midunits, hence $J(x) = S$, or $SxS = S$.

($\Leftarrow$) If $SxS = S$, the $J(x) = S$ i.e., $x \in J$, because there is only one maximal $\mathcal{J}$-class in S, as mentioned in Proposition 3-1.

3-5 Proposition. If $x \in J$, then $L(xv) = L(v)$, $R(vx) = R(v)$ and $vxv \in H_v$ for any $v \in M$.

Proof. From Proposition 3-4 we have that $SxS = S$. Then $SxS = SxvS$ since v is a midunit; thus $SxvS = S$, which implies that $xv \in J$ by Proposition 3-4. Hence we have $L(xv) \subset L(v) \subset J(xv)$, which implies that $L(xv) = L(v)$ by Lemma 3-3. Similarly we can prove $R(vx) = R(v)$. Hence it follows that $vxv \in H_v$.

3-6 Proposition. $xy \in J$ iff $x \in J$ and $y \in J$.

Proof. ($\Leftarrow$) If x and y are a pair of elements of J and v is a midunit, then vxv and vyv belong to H_v by proposition 3-5. From Proposition 3-1 we know that v is an idempotent so that H_v is a group. Since H_v is a group we have that $vxyv = vxvvyv$ is an element of H_v. On the other hand we have $SxyS = SvxyvS = SvS = S$; then $xy \in J$ by Proposition 3-4.

($\Rightarrow$) It follows easily from Proposition 3-4.

3-7 Proposition. $M = J \cap E$.

Proof. Let v be a midunit and u an element of $J \cap E$. It follows from Proposition 3-5 that $vuv \in H_v$, and also we have $(vuv)^2 = vuvvuv = vuvuv = vu^2v = vuv$, i.e. vuv is an idempotent in the group H_v. Since there is only one idempotent in a group we have $vuv = v$. From Lemma 2-3 we know that vu is a midunit. But since $(vu)u = vu$, one has that u is a midunit. This proves that $J \cap E \subset M$. On the other hand, $M \subset J \cap E$ follows from Proposition 3-1.

3-8 Proposition. J is a completely simple subsemigroup.

Proof. Since J is a $\mathcal{J}$-class in a compact semigroup hence J is a closed set in S. From Proposition 3-6 we have that J is a compact subsemigroup. If $K(J)$ is its minimal ideal and v is an idempotent in $K(J)$, then v is a midunit from Proposition 3-7. By assumption $S^2 = S$ so for each element z in J we can write z as $z = xy$ for some elements x and y in S. Thus

$$z = xy = xvy \in K(J), \text{ which implies } K(J) = J.$$

To prove the midunit v is primitive, we let $v \in J \cap E$, such that $vu = uv = u$. Then $v = (vu)v = (uv)v = uv = u$, which implies that v is primitive. Hence J is a completely simple subsemigroup [5].

We now collect the preceding results of this chapter into the main theorem of this chapter.

3-9 Theorem. Let S be a compact semigroup with midunit satisfying $S^2 = S$. Let J be the $\mathcal{J}$-class of the midunits and M be the set of all midunits of S. Then

(1) J is the unique maximal $\mathcal{J}$-class in S and is a completely simple subsemigroup, whose idempotents form a rectangular band.

(2) $M = J \cap E$.

(3) More precisely, let v be a midunit. Then Mv and vM are subsemigroups with left zero and right zero multiplications, respectively, and if $\psi: Mv \times H_v \times vM \to J$ is defined by $\psi((u,x,w)) = uxw$, then ψ is an isomorphism under which the image of $Mv \times \{v\} \times vM$ is M.

Proof. (1) and (2) follow from Proposition 2-4, Proposition 3-1, and Proposition 3-7. Analogous to the proof of Proposition 2-4, we can prove that Mv and vM are subsemigroups with left zero and right zero multiplication, respectively. From (1) and (2) we have $J = \cup \{H_v | v \in M\}$, so that the proof that ψ is an isomorphism is the same as that of Proposition 2-6.

CHAPTER IV

FINITE SEMIGROUPS WITH MIDUNIT

In this chapter we study finite semigroups which satisfy $S^2 = S$ with a unique maximal $\mathcal{J}$-class which is a completely simple subsemigroup. Recall that we proved in Theorem 3-9 that in a compact semigroup S with midunit satisfying $S^2 = S$, the $\mathcal{J}$-class of the midunits is maximal and is a completely simple subsemigroup. The converse of this is not true in general, even on a semigroup containing only five elements. That is, we will give an example of a semigroup S with five elements satisfying $S^2 = S$ which has a unique completely simple maximal $\mathcal{J}$-class, but does not have a midunit. Also we will give an example of a semigroup S with four elements satisfying $S^2 = S$ and containing a proper midunit. We begin with a generalized definition of midunit.

4-1 Definition. Let S be a semigroup and A and B be two nonempty subsets of S. A point v of S is called a midunit relative to [A,B] if avb = ab for all ordered pairs (a,b) of A × B.

A midunit, as previously defined, is then a midunit relative to [S,S].

4-2 Proposition. Let S be a semigroup satisfying $S^2 = S$ and v be an idempotent of S such that SvS = S. Then v is a midunit of S iff v is a midunit relative to $[vS\backslash vSv,\ Sv\backslash vSv]$.

Proof. ($\Rightarrow$) The proof is trivial.

($\Leftarrow$) If a and b are two elements of S, then there exists $a_1 . a_2$, b_1 and b_2 in S such that $a = a_1 v a_2$ and $b = b_1 v b_2$. In case $va_2 \in Sv$, i.e., $va_2 v = va_2$, then

$$avb = a_1 va_2 vb = a_1 (va_2 v) b = a_1 (va_2) b = ab.$$

In case $b_1 v \in vS$, then $avb = ab$, similarly. Suppose that neither of these cases occur, i.e., $va_2 \in vS \setminus vSv$ and $b_1 v \in Sv \setminus vSv$. Then by assumption that v is a midunit relative to $[vS \setminus vSv, Sv \setminus vSv]$, we have $(va_2) v (b_1 v) = va_2 b_1 v$; thus

$$avb = a_1 (va_2 vb_1 v) b_2 = a_1 (va_2 b_1 v) b_2 = ab.$$

Hence the proof is complete.

4-3 Proposition. Let S be a semigroup satisfying $S^2 = S$ and v be an idempotent of S such that $SvS = S$. Then the following statements hold.

(1) $vS \setminus vSv = \square$ iff v is a right identity for S,

(2) $Sv \setminus vSv = \square$ iff v is a left identity for S.

Proof. (1) ($\Leftarrow$) If v is a right identity, then $vS \setminus vSv = vS \setminus v(Sv) = vS \setminus vS = \square$. Similarly if v is a left identity for S, then $Sv \setminus vSv = \square$.

($\Rightarrow$) If $vS \setminus vSv = \square$, then $vS = vSv$. Therefore $S = S(vS) = SvSv = Sv$; i.e., v is a right identity for S. The proof of (2) is similar.

If A is a set, then its cardinality will be denoted by $n(A)$.

4-4 Proposition. Let S be a semigroup satisfying $S^2 = S$ and v is an idempotent of S such that $SvS = S$. If S is finite and v is neither a left nor a right identity of S, then

$$n(vSv) < n(S) - 1.$$

Proof. Suppose the conclusion is not true and let S be a finite semigroup with cardinality $n(S) = m$, and v be an idempotent such that $SvS = S$ and $n(vSv) \geq n(S) - 1 = m-1$. If v is neither a left identity nor a right identity we know that $vS \backslash vSv \neq \square$ and $Sv \backslash vSv \neq \square$ by Proposition 4-3. It is evident that $[vS \backslash vSv] \cap [Sv \backslash vSv] = \square$. Hence we have

$$\begin{aligned} n[vS \cup Sv] &= n(vS \backslash vSv) + n(Sv \backslash vSv) + n(vSv) \\ &\geq 1 + 1 + (m-1) = m + 1. \end{aligned}$$

That is a contradiction since $vS \cup Sv$ is a subset of S. Therefore

$$n(vSv) < n(S) - 1.$$

4-5 Proposition. Let S be a finite semigroup satisfying $S^2 = S$, and v be an idempotent of S such that $SvS = S$. Then the following statements hold.

(1) If $n(S) = 1, 2,$ or 3, then v is either a left or a right identity for S.

(2) If $n(S) = 4$, then v is a midunit for S.

Proof. (1) In the case $n(S) = 1$, it is evident that v is a identity for S. In the cases $n(S) = 2$ and $n(S) = 3$, we prove it as follows. If $S = K$, then there are only three possible multiplications on S; the group multiplication, left zero multiplication, or right zero multiplication. In case of left zero multiplication, every point is a right identity, in particular, v is a right identity. Similarly in the case of right zero multiplication, v is a left identity.

If the multiplication is the group multiplication, then v is the identity for S. Suppose $S \neq K$ and v is neither a left identity nor a right identity for S. According to Proposition 4-4 we have $n(vSv) = 0$ when $n(S) = 2$ and $n(vSv) = 1$ when $n(S) = 3$. But $n(vSv) = 0$ is impossible, since vSv is always a nonempty set. This implies that v is either a left or a right identity for S when $n(S) = 2$. For the case $n(S) = 3$ we obtain a contradiction as follows. Since $vSv \cap K \neq \square$ and $v \in vSv$, one has $v \in K$, which implies that $SvS = S = K$. That is again a contradiction since we assume $S \neq K$. Hence v must be a left or a right identity for S.

(2) The proof is immediate if $S = K$, so we assume that $S \neq K$. Then $v \notin K$, since otherwise $S = SvS \subset K$, which is a contradiction to the assumption $S \neq K$. If v is a right or a left identity for S, then v is a midunit. Hence we may assume that v is neither a left identity nor a right identity for S. Then by Proposition 4-3 we have $vS \backslash vSv \neq \square$ and $Sv \backslash vSv \neq \square$, i.e., $n(vS \backslash vSv) \geq 1$ and $n(Sv \backslash vSv) \geq 1$. It is claimed that $n(vSv) = 2$. For we know from Proposition 4-4 that $n(vSv) \leq 2$ and if $n(vSv) = 1$, then it implies that $v \in K$. It is easy to see that vSv, $vS \backslash vSv$ and $Sv \backslash vSv$ are mutually disjoint sets. Hence we have the following inequality:

$$4 = n(S) \geq n(vS \cup SV) = n(vSv) + n(vS \backslash vSv) + n(Sv \backslash vSv) \geq 4.$$

It follows that $n(vS \backslash vSv) = 1$ and $n(Sv \backslash vSv) = 1$. Since $v \in vSv$, $vSv \cap K \neq \square$, and $n(vSv) = 2$, we may write $vSv = \{v.k\}$ where $k = vkv \in K$. Also we let $vS \backslash vSv = \{a\}$ and $Sv \backslash vSv = \{b\}$. To prove v is a midunit for S, it is sufficient to prove that

v is a midunit relative to $[vS\backslash vSv, Sv\backslash vSv]$ by Proposition 4-2, i.e., to prove that $avb = ab$.

We prove first that either $av = k$ or $vb = k$. Otherwise, one has that $av = v$ and $vb = v$, since $\{av, vb\} \subset vSv$ and $vSv = \{v,k\}$. Then we have $b(av) = bv = b$. In order to arrive at a contradiction we consider the element ba. There are only four possibilities that is $ba = v$, $ba = a$, $ba = b$, or $ba = k$. In the case $ba = v$, we have $b = bv = b(av) = (ba)v = v$, which is a contradiction since $b \neq v$. In the case $ba = k$, we have $v = v^2 = (vb)(av) = v(ba)v = vkv = k$, which is a contradiction since $v \neq k$. Hence we may assume $ba = a$. Then $(ba)v = av = v$ and $b(av) = bv = b$, which implies that $b = v$. That is impossible since $b \neq v$. Similarly if $ba = b$, then $v(ba) = vb = v$ and $(vb)a = va = a$, which is again a contradiction since $a \neq v$. Hence it is true that either $av = k$ or $vb = k$. Therefore, we have $avb = ak$ or $avb = kb$. Then $avb = k$ since either $ak = v$ or $kb = v$ will imply that $v \in K$, and hence $S = K$.

In order to prove $ab = k$, let us assume that $ab = v$, since $ab \in vSv$ and $vSv = \{v,k\}$. It follows that $\{a,b\} \subset J$, where J is the $\mathcal{J}$-class of the $\mathcal{J}$-maximal element v in S. Thus both a and b are $\mathcal{J}$-maximal elements of S. By Theorem 1-6, there is an element x of S such that $ax = a$ and an element y of S such that $yb = b$. In the case $x = v$, then $av = a$ implies that $a \in vSv$. It is a contradiction since $a \notin vSv$. In the case $x = b$, then $ab = a$, contradicting our assumption $ab = v$. Further, if $x = k$, then $ak = a$, contradicting the assumption that $v = ab$, since $v = ab = akb = k$. Hence the only possible case will be $x = a$ which implies that $a^2 = a$. This implies that $ava = a$, where $av \in vSv$. It follows that $av = v$, since $vSv = \{v,k\}$ and if $av = k$, then $k = av = a(ab) = a^2b = ab = v$. Similarly, one can check that the only possibility for y is $y = b$ so that $b^2 = b$ and hence $vb = v$. But by the preceding argument we know it is impossible that $av = v$ and $vb = v$. Therefore, we proved that $ab = k$. The proof is complete.

4-6 Example. We give two examples of semigroups of four elements with midunit satisfying $S^2 = S$. The first one is an example of a semigroup S with zero and a proper midunit v satisfying $SvS = S$. Let $S = \{v,a,b,0\}$ with multiplication as follows.

Table 4-1

.	v	a	b	0
v	v	a	0	0
a	0	0	0	0
b	b	0	0	0
0	0	0	0	0

It is easy to see that $vSv = \{v,0\}$, $vS = \{v,a,0\}$, $Sv = \{v,b,0\}$, and $SvS = S$. It is evident that b is neither a left identity nor a right identity for S, since vS and Sv both are proper subsets of S. To check that v is a midunit for S it is sufficient to check that v is a midunit relative to $[vS\backslash vSv, Sv\backslash vSv]$. We see that $vS\backslash vSv = \{a\}$ and

$Sv\backslash vSv = \{b\}$. While $ab = 0 = avb$ according to the multiplication table. Hence v is a proper midunit for S.

The second example is of a semigroup satisfying $S = K$ in which every point is a proper midunit. We construct this example as follows. Let $S_1 = \{a,b\}$ with left zero multiplication and $S_2 = \{c,d\}$ with right zero multiplication, and we define $S = S_1 \times S_2$ with coordinatewise multiplication. Then it is not difficult to check that $S = K$ and each point in S is a proper midunit for S.

We end this chapter with an example of a semigroup with five elements which satisfies the condition $SvS = S$ for an idempotent v and has a unique maximal $\mathcal{J}$-class which is a completely simple subsemigroup, but S does not have a midunit.

4-7 Example. Let $S = \{v,a,b,c,0\}$ with multiplication as in the following table.

Table 4-2

	v	a	b	c	0
v	v	a	b	a	0
a	a	0	0	0	0
b	a	0	0	a	0
c	c	0	0	0	0
0	0	0	0	0	0

Then it is easy to see that S is a semigroup, v is an idempotent, $J = \{v\}$ is the unique maximal $\mathcal{J}$-class in S which is a completely simple subsemigroup. Also we have that

$SvS = S$, for $vS = \{v,a,b,0\}$ and $Sv = \{v,a,c,0\}$. But v is not a midunit because $bc = a$ while $bvc = 0$.

CHAPTER V

THE CODIMENSION OF THE $\mathcal{J}$-CLASS OF MIDUNITS IN A CONTINUUM SEMIGROUP

This chapter is devoted to proving that the $\mathcal{J}$-class J of the midunits in a continuum semigroup of codimension one with midunit satisfying $S^2 = S \neq K$ is totally disconnected. Recall that a compact space X has codimension (abbreviated "cd") less than or equal to a nonnegative integer n if $H^n(X) \xrightarrow{i^*} H^n(A)$ is epimorphism for each closed subset A of X, where i is the inclusion map $i: A \to X$. This is equivalent to the statement that $H^n(B) \xrightarrow{k^*} H^n(A)$ is epimorphism for each pair of closed subsets A and B of X with $A \subset B$, where k is the inclusion map $k: A \to B$. We define $cd\Box = -1$, and $cd(X) = \inf\{n | cd(X) \leq n\}$ for a nonempty compact space X [3]. Recall that a floor for a nonzero element $h \in H^n(X)$ is a closed subset F of X such that $h|F \neq 0$ while $h|F' = 0$ for any closed proper subset F′ of F. If X is compact Hausdorff and $0 \neq h \in H^p(X)$, then h has a floor. Moreover, any floor is connected [13]. We proceed with a theorem due to Wallace [20].

5-1 Theorem. Let S be a continuum semigroup with $cd(S) \leq n$ and A be a closed subset of S which is a floor for some $0 \neq h \in H^n(A)$. It $t_1A = A$ ($At_1 = A$) for some $t_1 \in S$, then $tA = A$ ($At = A$) for each $t \in S$.

The following lemma is a result concerning $\mathcal{H}$-classes in compact semigroups.

5-2 Lemma. Let S be a compact semigroup and b be a point of S. If H_b is not a singleton set, then there exists an idempotent e such that $H_b = eH_b$.

Proof. If d is an element in H_b which is different from b, then we have $b = xd$ and $d = yb$ for some x and y in S. This implies that $b = xyb$; thus $b = eb$ for some idempotent e, from the Swelling Lemma 1-5. If z is an element of H_b other than b, then there is a point u in S such that $z = bu$, then $ez = ebu = bu = z$, i.e., $H_b = eH_b$.

It is well known that if S is a compact semigroup then each $\mathcal{H}$-class is homeomorphic to a topological group [6; p.35]. Also if X is a compact topological group with codimension n and $\mathbb{Z}$ is the additive group of integers then $H^n(X;\mathbb{Z}) \neq 0$ [6;p.54]. Consequently we have that if S is a compact semigroup and H_a is an $\mathcal{H}$-class of a in S with $cd(H_a) = n$, then $H^n(H_a;\mathbb{Z}) \neq 0$.

5-3 Theorem. If S is a continuum semigroup with finite codimension n, then $cd(H_a) < cd(S)$ for all a in S such that $H_a \neq K$.

Proof. We prove first that $cd(H_a) < cd(S)$ for each $a \in S\backslash K$. Suppose on the contrary that there exists an element $a \in S\backslash K$ such that $cd(H_a) = n$. If H_a is a singleton set, then $cd(H_a) = 0$, which implies that $cd(S) = 0$; i.e., S is a totally disconnected space. But this is impossible since S is a continuum with more than one element. Lemma 5-2 guarantees the existence of an idempotent e in S such that $H_a = eH_a$. Let h be a nonzero element of $H^n(H_a;Z)$ and let A be a floor for h. Since $A \subset H_a$ we have $A = eA$. Therefore $tA = A$ for each $t \in S$ by Theorem 5-1. Since $a \in S \setminus K$, we have that $H_a \cap K = \square$, which implies that $A \cap K = \square$. If we let t_0 be a point of K, then we have $A = t_0 A \subset K$, since K is an ideal in S. It is a contradiction since we know that $A \cap K = \square$.

We next prove that $cd(H_a) < cd(S)$ for each $a \in K$ with $H_a \neq K$. Suppose on the contrary that $cd(H_a) = n$ for some $a \in K$ with $H_a \neq K$. Then K consists of at least two $\mathcal{H}$-classes. We know that the $\mathcal{H}$-class in K are mutually homeomorphic groups, so that we have $cd(H_u) = n$ for each $u \in K \cap E$. In order to obtain a contradiction, we let e and f be distinct idempotents in K. This is possible since K contains at least two $\mathcal{H}$-classes. As before, we let h be a nonzero element of $H^n(H_e;Z)$ and let A be a floor for h. Since $A \subset H_e$ is a group with identity e, it follows that $A = eA$. By Theorem 5-1 we know that $tA = A$ for each $t \in S$. This implies that $e \in A$ since A is a subset of the group H_a while S contains all possible inverses in H_e of A. Therefore $fe \in H_e$, since $tA = A$ for each $t \in S$. Similarly we could prove that $fe \in H_f$. But this is a contradiction since H_e and H_f are mutually disjoint. Hence the proof is complete.

5-4 Proposition. Let X and Y be two compact Hausdorff spaces. If $cd(X)$ is finite and Y is totally disconnected, then $cd(X \times Y) = cd(X)$.

Proof. Suppose $cd(X \times Y) = n$ and let A be a closed subset of $X \times Y$ such that the homomorphism induced by the inclusion map from A into $X \times Y$, i.e., $i^*: H^{n-1}(X\times Y) \to H^{n-1}(A)$, is not epimorphic. Hence there exists a nonzero element $h \in H^{n-1}(A)$ which is not extendible to $X \times Y$. According to Theorem 1-3, we let R be a roof for h and consider $h_0 = h|R \cap A \neq 0$. Then R is also a roof for h_0 and R is connected. Let π_1 and π_2 be the two projection maps from $X \times Y$ to X and Y, respectively, i.e., $\pi_1((x,y)) = x$ and $\pi_2((x,y)) = y$. Let j be the inclusion map from R into $X \times Y$, and let η be defined by $\eta = \pi_1 o j: R \to X$. We want to prove that η is injective. Suppose not, and let (a,b) and (a,c) be two points in R such that $b \neq c$. Then $\pi_2(R)$ is a nondegenerate connected set in Y, since R is connected, π_2 is a map and $\pi_2((a,b)) \neq \pi_2((a,c))$. This contradicts the assumption that Y is totally disconnected. Then η is an injective map. Consider the following diagram:

$$\begin{tikzcd}
R \arrow[r, "\eta"] & \eta[R] \\
R \cap A \arrow[u, "k"] \arrow[r, "\eta|R\cap A"] & \eta[R \cap A] \arrow[u, "\ell"']
\end{tikzcd}$$

where k is the inclusion map from $R \cap A$ into R, and ℓ is the inclusion map from $\eta[R \cap A]$ into $\eta[R]$. Also η and $\eta|R \cap A$ are homeomorphisms since η is an injective map.

Then the following diagram is commutative, i.e., $\eta^* o k^* = (\eta|R \cap A)^* \circ \ell^*$.

$$\begin{tikzcd}
H^{n-1}(R) \arrow[d, "k^*"'] & H^{n-1}(\eta[R]) \arrow[l, "\eta^*"'] \arrow[d, "\ell^*"] \\
H^{n-1}(R \cap A) & H^{n-1}(\eta[R \cap A]) \arrow[l, "(\eta|R\cap A)^*"']
\end{tikzcd}$$

We know that both η^* and $(\eta|R\cap A)^*$ are isomorphisms, and k^* is not an epimorphism, so ℓ^* is not an epimorphism. This implies that $cd(X) > n-1$. But we know that $cd(X) \leq cd(X \times Y) = n$, therefore $cd(X) = cd(X \times Y)$. Hence the proof is complete.

5-5 Corollary. Let S be a compact semigroup with midunit of finite codimension satisfying $S^2 = S \neq K$. Let J be the $\mathcal{J}$-class of the midunits, M be the set of all midunits and H_v be the $\mathcal{H}$-class of midunit v. If H_v is totally disconnected for some midunit, then cdJ = cdM.

Proof. It is not difficult to prove from Theorem 3-9 that J is homeomorphic to the space $M \times H_v$. The conclusion follows from Proposition 5-4.

It has been proved by Cohen and Koch [4] that if A is a closed subset of a compact Hausdorff space X with $cd(X) \leq n$, then $cd(X/A) \leq n$ [4]. Also they proved in the same paper that if S is a continuum semigroup satisfying S = ESE with zero then $H^1(S) = 0$.

5-6 Theorem. Let S be a continuum semigroup of codimension one with midunit satisfying $S^2 = S \neq K$. Then the $\mathcal{J}$-class J of midunits is totally disconnected.

Proof. As before we let M be the set of all midunits and H_v be the $\mathcal{H}$-class of the midunit v. From Theorem 5-3 we know that H_v is totally disconnected for each midunit v. In order to prove that J is totally disconnected it suffices by Corollary 5-5 to prove that M is totally disconnected. In case K is not a singleton set we may consider $\overline{S} = S/K$, the Rees quotient of K. Then $\overline{S}$ is a continuum semigroup with zero and $\mathrm{cd}(\overline{S}) \leq 1$. If we let $\psi: S \rightarrow S/K$ be the natural homomorphism, then $\psi(v)$ is a midunit for S and $\psi(J)$, the $\mathcal{J}$-class of $\psi(v)$ in $\overline{S}$, is homeomorphic to J. Hence without loss of generality we may assume $K = \{0\}$. Consider $T = MSM$, then T is a nondegenerated subcontinuum subsemigroup with zero and $E(T)TE(T) = T$ since $M \subseteq E(T)$. Also we know that $\mathrm{cd}(MSM) \leq 1$ and $H^1(MSM) = 0$.

Let C be a component in M with more than one point and u and v be two different points in C. Then $vSv \cup uSu$ is a closed subset in MSM and $C \cap [vSv \cup uSv] = \{v,u\}$. It follows from the Mayer-Vietoris Sequence Theorem 1-1 that the following sequence is exact,

$$\tilde{H}^0(C) \times \tilde{H}^0(vSv \cup uSu) \xrightarrow{i^*-i'^*} \tilde{H}^0(\{u,v\}) \xrightarrow{\Delta} H^1(C \cup [vSv \cup uSu]).$$

If G is the nontrivial coefficient group for the cohomology, then $\tilde{H}^0(\{u,v\}) = G$ while $\tilde{H}^0(C) = 0$ and $\tilde{H}^0(C \cup [vSv \cup uSu]) = 0$, since both C and $C \cup [vSv \cup uSu]$ are connected. Therefore Δ is a monomorphism, which implies that $H^1(C \cup [vSv \cup uSu]) \neq 0$. This is a contradiction since $H^1(MSM) = 0$, $\mathrm{cd}(MSM) \leq 1$ and $C \cup [vSv \cup uSu]$ is a closed subset of MSM. Hence the proof is complete.

CHAPTER VI

THE COMPLEMENT OF THE $\mathcal{J}$-CLASS OF MIDUNITS AND MINIMAL CONTINUUM SEMIGROUPS WITH MIDUNIT

The first part of this chapter is devoted to a study of the structure of the complement of J in S, where S is a continuum semigroup with midunit satisfying $S^2 = S \neq K$ and J is the $\mathcal{J}$-class of the midunits of S. We prove that the complement of J is a dense connected maximal ideal in S and no subset of J cuts S. If, furthermore, S is locally connected and U is an open set containing J, then there exists an open set V in U containing J such that S\V is a connected ideal. The second part of this chapter investigates the structure of minimal continuum semigroups with midunit satisfying $S^2 = S \neq K$ (see Definition 6-5) and their relation to irreducible clans. We proceed with a proposition that is due to Koch and Wallace [8]. Since our proof is slightly different from theirs we present it here.

6-1 Proposition. Let S be a continuum semigroup satisfying $S^2 = S \neq K$ with midunit and J be the $\mathcal{J}$-class of the midunits. Then S\J is a dense connected maximal ideal of S.

Proof. From Proposition 3-6 and Theorem 3-9 it follows that S\J is a maximal ideal in S. Since the closure of ideal is again an ideal hence [S\J]* is an ideal. From the maximality of $S\backslash J$ it follows that either $[S\backslash J]^* = S$ or $[S\backslash J]^* = S\ J$. If $S\backslash J = [S\backslash J]^*$, then J is a proper nonempty closed and open set of S which is impossible because S is connected. Hence $[S\backslash J]^* = S$, $S\backslash J$ is dense in S. It remains to show that $S\backslash J$ is connected. Since $S^2 = S$, for each $z \in S\backslash J$ there exists a pair of elements a_z and b_z of S such that $z = a_z b_z$. From Proposition 3-6 we know that for each $z \in S\backslash J$, $a_z S b_z$ is disjoint from J and $z \in a_z S b_z$ since S contains a midunit.

Then $S\backslash J = \cup\{a_z S b_z \mid z \in S\backslash J\}$ is a union of connected sets, so since each $a_z S b_z$ meets K, we have that $S\backslash J$ is a connected set. The proof is complete.

Let A and B be two subsets of a space X. It is known in general topology that if A is connected and $A \subset B \subset A^*$, then B is connected. Then the following proposition can be proved easily.

6-2 Proposition. Let S be a continuum semigroup with midunit satisfying $S^2 = S \neq K$ and J be the $\mathcal{J}$-class of the midunits. Then no subset of J cuts S.

Proof. Let N be a subset of J, then $S\backslash J \subset S\backslash N \subset S$. From Proposition 6-1 we know that $S\backslash J$ is connected and $[S\backslash J]^* = S$. Therefore $S\backslash N$ is connected.

6-3 Proposition. Let S be a continuum semigroup with midunit satisfying $S^2 = S \neq K$ and J be the $\mathcal{J}$-class of the midunits. If S is locally connected and U is an open set containing J, then there exists an open set V containing J with $V \subset U$ such that $S\backslash V$ is a connected ideal of S.

Proof. Let $A = S\backslash U$ and for each $a \in A$ let $N(a)$ be a closed connected neighborhood which is small enough that it is disjoint from J. Since A is compact, we let $N(a_1)$, $N(a_2), \ldots, N(a_m)$ be a finite cover of A. For each a_j let b_j and c_j be a pair of elements of S such that $a_j = b_j c_j$. Then $b_j S c_j$ is a subcontinuum which contains a_j and meets K for each $j = 1,2,\ldots,m$. Let

$$B = [\bigcup_{i=1}^{m} N(a_i)] \cup [\bigcup_{i=1}^{m} b_i S c_i] \cup K.$$

Then B is a subcontinuum which contains A and is disjoint from J. From Proposition 3-6 we know that $D = B \cup BS \cup SB \cup SBS$ is a subcontinuum ideal of S and is disjoint from J. Now we simply let $V = S \backslash D$. It is then easy to see that $J \subset V \subset U$ and $S \backslash V = D$. This completes the proof.

6-4 Definition. Let S be a continuum semigroup with midunit satisfying $S^2 = S \neq K$ and M be the set of all midunits of S. Then S is called minimal if it does not contain a proper subcontinuum subsemigroup T with $M \subset T$ and $T \cap K \neq \square$.

6-5 Theorem. Let S be a continuum semigroup with midunit satisfying $S^2 = S \neq K$ and M be the set of all midunits. Then S contains a continuum subsemigroup T such that $M \subset T$, $T \cap K \neq \square$ and T is minimal.

Proof. This follows immediately through an application of Zorn's Lemma.

Recall that a clan S with identity v is called irreducible if it does not contain a proper subcontinuum semigroup T containing v and satisfying $T \cap K \neq \square$. In an irreducible clan with identity v, one has $H_v = \{v\}$ [6; p. 156].

6-6 Theorem. Let S be a continuum semigroup satisfying $S^2 = S$ with zero and midunit. M be the set of all midunits, J be the $\mathcal{J}$-class of midunits and T be minimal continuum subsemigroup such that $M \subset T$, and $0 \in T$. Then the following statements hold.

(1) $MTM = T$.

(2) vTv is irreducible for each $v \in M$.

(3) $T \cap J = M$.

(4) Consider the cartesian product $Mv \times vTv \times vM$ with coordinatewise multiplication, and define $\psi\colon Mv \times vTv \times vM \to T$ by $\psi((u,x,w)) = uxw$. Then ψ is a surjective homomorphism.

Proof. (1) $MTM \subset T$ follows from the facts that $M \subset T$ and T is a subsemigroup. On the other hand, since MTM is a continuum subsemigroup such that $M \subset MTM$ and MTM contains 0, we have by the minimal condition of T that $T \subset MTM$. Hence $T = MTM$.

(2) Suppose not, i.e., there exists a point $v \in M$ such that vTv is not irreducible. Hence there exists a subclan of vTv, say T_0, such that T_0 is a proper subset of vTv; $v \in T_0$ and $0 \in T_0$. Then MT_0M is a continuum subsemigroup of S such that $M \subset MT_0M$, since $v \in T_0$ and $MvM = M^2 = M$, and $0 \in MT_0M$. Since T is minimal we have $MT_0M = T$, which implies that $vMT_0Mv = vTv$. Since $vMT_0Mv = vT_0v$, we have that $vT_0v = T_0 = vTv$, which contradicts the assumption that T_0 is a proper subset of vTv. This completes the proof.

(3) It is sufficient to prove that $T \cap J \subset M$. Since by (2) vTv is irreducible for each $v \in M$, we know that

$vTv \cap H_v = \{v\}$, where H_v is the maximal group containing v. If $z \in T \cap J$ then $z = vzv$ for some $v \in M$, since by Theorem 3-9 we know that J is the union of all maximal group H_w with $w \in M$. This implies that $z \in H_v \cap vTv$. Then $z = v$ and the proof is complete.

(4) Since $MTM = T$, each $z \in T$ can be written as $z = uzw$ for some midunits u and w. On the other hand since $\psi((uv, vzv, uw)) = uvvzvvw = uzw = z$, then ψ is surjective.

Also since $\psi[(u,x,w)(\overline{u},\overline{x},\overline{w})] = \psi[(u\overline{u}, x\overline{x}, w\overline{w})] = ux\overline{xw}$ and $\psi((u,x,w))\ \psi((\overline{u},\overline{x},\overline{w})) = (uxw)(\overline{uxw}) = ux\overline{xw}$ for any pair of elements (u,x,w) and $(\overline{u},\overline{x},\overline{w})$ of $Mv \times vTv \times vM$, we have that ψ is a surjective homomorphism.

CHAPTER VII

A METHOD OF CONSTRUCTING SEMIGROUPS WITH MIDUNIT

McCharen has pointed out a method of constructing new semigroups with midunit from given known semigroups [11]. It is our purpose in this chapter to introduce this method and discuss some basic properties of it. In particular, we investigate those algebraically defined sets which are invariant under this construction. The importance of this method to this work is that it could transfer problems in general semigroups into problems of semigroups with midunit and vice-versa. We prove in this way that if S is a compact semigroup, v is a $\mathcal{J}$-maximal idempotent, J is the $\mathcal{J}$-class of v, and $vJv \subset J$, then J is homeomorphic to a cartesian product of a rectangular band with H_v, the maximal subgroup which contains v. Furthermore, if $cd(S) = 1$ and $S \neq K$, then J is totally disconnected.

7-1 Proposition. Let S be a semigroup and v be an idempotent in S. We define a multiplication "*" on S by

$$a*b = avb, \text{ for all } a \text{ and } b \text{ of } S.$$

Then the following statements are true.

(1) $(S;*)$ is a semigroup under the multiplication "*", and v is an idempotent midunit in $(S;*)$.

(2) $S*S = S$ iff $SvS = S$.

Proof. (1) It is routine to check the associativity of the multiplication "$*$", because S is a given semigroup. To prove v is a midunit, we let x and y be a pair of elements of $(S;*)$. Then

$$x*v*y = (x*v)*y = (xvv)*y = (xv)*y = (xvv)y = xvy = x*y,$$

which implies that v is a midunit of $(S;*)$. Also $v*v = vvv = v$, i.e., v is an idempotent in $(S;*)$.

(2) The proof of this part is trivial.

It is not difficult to prove that if N is a right (left, two-side) ideal of S, then N is a right (left, two-sided) ideal of $(S;*)$.

7-2 Proposition. The following two statements are equivalent:

(1) N is a minimal right (left, two-sided) ideal in S.

(2) N is a minimal right (left, two-sided) ideal in $(S;*)$.

Proof. We only prove the case when N is a minimal right ideal. The other cases can be proved analogously.

(1) ⇒ (2) It is not difficult to prove that N is a right ideal in $(S;*)$. In order to prove that N is a minimal right ideal in $(S;*)$, we let P be a right ideal in $(S;*)$ such that $P \subset N$. Then we have that $PvS = P*S \subset P \subset N$. Since PvS is a right ideal of S contained in the minimal right ideal N, we have that $PvS = P = N$. This shows that N is a minimal right ideal of $(S;*)$.

(2) ⇒ (1) Since N is a right ideal of $(S;*)$, we have that $NvS = N*S \subset N$. Also it is easy to see that NvS is a right ideal of $(S;*)$ which is contained in the minimal right ideal N of $(S;*)$. Therefore $N = NvS$, so that N is a right

ideal of S. In order to prove that N is a minimal right ideal in S, we let P be a right ideal in S such that $P \subset N$. Then $PS \subset P \subset N$. On the other hand, PS is a right ideal of $(S;*)$, so that by the minimality of N we have $PS = P = N$, which proves that N is a minimal right ideal of S.

As usual, for an element z in S, we let $L(z)$, $(R(z), J(z))$ be the smallest left (right, two-sided) ideal of S which contains z, and $\overline{L}(z)$, $(\overline{R}(z)$, $\overline{J}(z))$ be the smallest left (right, two-sided) ideal of $(S;*)$ which contains z. Also we let H_v be the maximal group in S which contains v, and let $\overline{H}_v$ be the maximal group in $(S;*)$ which contains v.

7-3 Proposition. If z is an element of S, then the following statements hold.

(1) $\overline{L}(z) \subset L(z)$, $\overline{R}(z) \subset R(z)$, and $\overline{J}(z) \subset J(z)$

(2) $\overline{L}(vz) = L(vz)$, $\overline{R}(zv) = R(zv)$, and $\overline{J}(vzv) = J(vzv)$

(3) $\overline{J}(v) = J(v)$, and $H_v = \overline{H}_v$

Proof. (1) Since $\overline{L}(z) = z \cup S*z = z \cup Svz \subset L(z)$, we have that $\overline{L}(z) \subset L(z)$. We can prove that $\overline{R}(z) \subset R(z)$ and $\overline{J}(z) \subset J(z)$.

(2) Since $\overline{L}(vz) = vz \cup S*(vz) = vz \cup Svvz = vz \cup Svz = L(z)$ we have that $L(vz) = L(z)$. We can prove analogously that $\overline{R}(zv) = R(zv)$ and $\overline{J}(vzv) = J(vzv)$.

(3) It follows from (2) that $\overline{J}(v) = J(v)$, $\overline{L}(v) = L(v)$ and $\overline{R}(v) = R(v)$, which implies that $\overline{H}_v = H_v$.

7-4 Proposition. Let S be a semigroup and v be an idempotent of S such that $SvS = S$. Let "$*$" be the multiplication defined on S by $a*b = avb$ for all a and b of S. If we let J be the

$\mathcal{J}$-class of v in S and $\bar{J}$ be the $\mathcal{J}$-class of v in (S;*) and if vJv $\subset$ J, then $\bar{J}$ = J.

Proof. In order to prove that $\bar{J} \subset J$, we let $z \in \bar{J}$. Thus we have $v \in z \cup z{*}S \cup S{*}z \cup S{*}z{*}S$, which implies that $S{*}v{*}S \subset S{*}z{*}S$. But $S{*}v{*}S = SvS = S$ by assumption, which implies that $S{*}z{*}S = S$, i.e., $SvzvS = S$. Therefore $z \in J$ since $SvzvS \subset SzS$. In order to prove that $J \subset \bar{J}$, we let $z \in J$. Since $vJv \subset J$, we have that $vzv \in J$; that is, $SvzvS = S$. On the other hand, $S{*}z{*}S = SvzvS = S$, which implies that $z \in \bar{J}$. Therefore the proof is complete.

7-5 Theorem. Let S be a compact semigroup with minimal ideal K. Let v be a $\mathcal{J}$-maximal idempotent of S such that its $\mathcal{J}$-class J satisfies $vJv \subset J$ and let "*" be the multiplication defined on S by $a{*}b = avb$ for all a and b of S. Then the following statements hold.

(1) J is homeomorphic to a cartesian product of a rectangular band and H_v, the maximal group which contains v. More precisely, (J;*) is the cartesian product of a rectangular band and H_v with coordinatewise multiplication.

(2) If $cd(S) = 1$ and $S \neq K$, then J is totally disconnected.

Proof. Let T be defined by T = SvS, then T is a compact semigroup satisfying TvT = T and contains v as a $\mathcal{J}$-maximal idempotent in T. Let $\bar{J}$ be the $\mathcal{J}$-class of v in T. It is claimed that $\bar{J} = J$. In order to prove that $\bar{J} \subset J$, let t be an element of $\bar{J}$. Then we have $\bar{J}(t) = \bar{J}(v) = TvT = T = SvS$. It follows that $J(v) = SvS \subset J(t)$, since $\bar{J}(t) \subset J(t)$. By the maximality of v in S we obtain $J(t) = J(v)$; i.e., $t \in J$. Therefore $\overline{J} \subset J$. In order to prove that $J \subset \overline{J}$, let z be an element of J. By the assumption $vJv \subset J$, it follows that $vzv \in J$. This implies that $J(vzv) = J(v) = SvS$. Also it can easily be proved that $J(vzv) = \overline{J}(vzv)$ and $\overline{J}(vzv) \subset \overline{J}(z)$. Therefore, we have that $T = SvS = J(vzv) = \overline{J}(vzv) \subset \overline{J}(z) \subset J$ and hence that $\overline{J}(z) = T$ or $z \in \overline{J}$. Hence we have proved that $J = \overline{J}$. If we let $\overline{H}_v$ be the maximal subgroup in T which contains v, then it is evident that $\overline{H}_v = H_v$.

If $\overline{\overline{J}}$ is the $\mathcal{J}$-class of v in $(T;*)$, then by Theorem 3-9, $(\overline{\overline{J}};*)$ is isomorphic to a cartesian product of a rectangular band and H_v with coordinatewise multiplication. By Proposition 7-4 we have $\overline{\overline{J}} = J$, since $v\overline{J}v \subset \overline{J}$ follows directly from the fact that $J = \overline{J}$ and the assumption $vJv \subset J$. Consequently, we have proved part (1).

Since $cd(S) = 1$ and $S \neq K$ imply that $cd(T) \leq 1$ and $T \neq K(T)$, the conclusion of part (2) follows easily from Proposition 7-2 and Theorem 5-6.

CHAPTER VIII

SEMIGROUPS ON D-SPACES

In this chapter we are seeking sufficient conditions on a semigroup S under which S must have either a left or a right identity and consequently S does not admit a proper midunit. We show that the algebraic condition $S^2 = S \neq K$ and the topological condition that S be D-space (to be defined in Definition 8-6) are one such set of conditions. Recall that a midunit v of a semigroup S is proper if it is neither a left identity nor a right identity of S. In Example 2-1 we defined two multiplications on a closed unit interval, in which a midunit exists without being a left identity, right identity, or even an idempotent. It seems difficult to study the notion of midunit without some algebraic restriction. One algebraic condition, which neither of the two multiplications given in Example 2-1 satisfies, is $S^2 = S \neq K$. It is not difficult to prove the following proposition.

8-1 Proposition. Let S be topologically the closed unit interval [0,1] and satisfy $S^2 = S \neq K$. Then there exists a point v of S which is either a left or a right identity of S and is located at one endpoint of [0,1].

This example motivated us to seek a more general topological condition under which a semigroup S satisfying $S^2 = S \neq K$ must have either a left or right identity.

8-2 Definition. Let X be a continuum. A point p of X is a D-point iff for any two subcontinuua C_1 and C_2 with $p \in C_1 \cap C_2$. either $C_1 \subset C_2$ or $C_2 \subset C_1$.

Let X be a continuum with D-point p. For each point a of X, let $\mathfrak{F}_a$ be the collection of all subcontinuua of X which contains both a and p. Then $\mathfrak{F}_a$ is a non-empty collection of compact connected subsets of X and $\mathfrak{F}_a$ is totally ordered by inclusion. If $L[a] = \cap \mathfrak{F}_a$, then L[a] is the minimal subcontinuum containing a and p. It is easy to see that $L[p] = \{p\}$, since $\{p\}$ is itself a subcontinuum which contains p. We define a relation $\leq$ on X as $a \leq b$ iff $L[a] \subset L[b]$. Since L[a] and L[b] are always comparable under inclusion for any pair of elements a and b of X, $\leq$ is a total quasi-order on X. By the definition of $\leq$ and construction of L[a] we have $L[a] = \{b | b \leq a\}$, and call L[a] the lower set of a in X. The sets $U[a] = \{b | a \leq b\}$ and $L_a = L[a] \cap U[a]$ are called the upper set and level set of a respectively. For convenience, we write $a \approx b$ iff $a \leq b$ and $b \leq a$, and $a < b$ iff $a \leq b$ and $b \not\leq a$. Recall that a quasi-order $\leq$ on a set X is called order dense if and only if for any pair of elements a and b of X satisfying $a < b$, there exists a point c of X such that $a < c < b$.

8-3 Lemma. Let X be a continuum with a D-point p. If U[a] is closed for each point a of X, then $\leq$ is order dense.

Proof. Suppose not; i.e., suppose there exists a pair of elements a and b in X, with $a < b$ and no point c in X satisfies $a < c < b$. Since $\leq$ is a total quasi-order on X we have

$L[a] \cup U[b] = X$ and $L[a] \cap U[b] = \square$. Then X is a union of a pair of disjoint nonempty closed subsets $L[a]$ and $U[b]$, which contradicts the assumption that X is connected. Hence the proof is complete.

8-4 Proposition. Let X be a continuum with a D-point p and $\le$ be the induced quasi-order. Then the following two statements are equivalent.

(1) "$\le$" is a closed relation on X, and

(2) $U[a]$ is closed for each point a of X.

Proof. (2) $\Rightarrow$ (1) Let b and c be a pair of elements of X such that $(b,c) \notin \le$. Since $\le$ is a total quasi-order on X we have $c < b$. By Lemma 8-3 there is a point d in X satisfying $c < d < b$. If $U = X \setminus L[d]$ and $V = X \setminus U[d]$, then U is an open set containing b while V is an open set containing c. It is claimed that $(U \times V) \cap \le = \square$. Suppose not; i.e., suppose there exists a pair of elements x and y of X such that $(x,y) \in (U \times V) \cap \le$. Then $d < x, y < d$ and $x \le y$, which implies that $d < d$. But this is a contradiction. Hence the proof is complete.

(1) $\Rightarrow$ (2) We omit the proof because it is trivial.

8-5 Remark. There exists a continuum with a D-point whose induced quasi-order $\le$ is not closed. Let X be a space defined by

$$X = \{(x,y) \mid y = \sin\left(\frac{1}{x+\frac{1}{\pi}}\right), -\frac{1}{\pi} < x \le 0\} \cup$$

$$\{(x,y) \mid y = \sin\left(\frac{1}{\frac{1}{\pi} - x}\right), 0 \le x < \frac{1}{\pi}\} \cup$$

$$\{(x,y) \mid -1 \le y \le 1, x = \frac{1}{\pi} \text{ or } x = -\frac{1}{\pi}\},$$

with the usual topology as seen in the following figure.

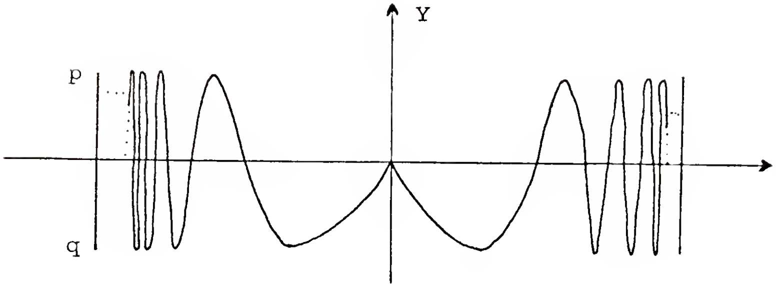

Figure 8-1

Then X is a continuum with a D-point $p = (-\frac{1}{\pi}, 1)$. But the induced quasi-order $\le$ is not closed, since $U[q] = X \backslash \{(x,y) \mid x = -\frac{1}{\pi}, -1 \le y \le 1\}$ is not a closed set in X where $q = (-\frac{1}{\pi}, -1)$. By Proposition 8-1 we know that $\le$ is not a closed relation.

8-6 Definition. A continuum X with a D-point p is called a D-space if the induced quasi-order $\le$ is a closed relation on X.

We have seen in Remark 8-5 that X is not locally connected at p, while the induced quasi-order $\le$ is not closed. But if X is a D-space, then X is locally connected at its D-point p.

8-7 Proposition. If X is a D-space with D-point p, then X is locally connected at p.

Proof. Let V be an open proper subset containing the D-point p. If $\mathcal{L}$ is the collection of all $L[x] \cap (X \backslash V)$ for each x of $X \backslash V$, then $\mathcal{L}$ is a nonempty collection of nonempty closed subsets of X and $\mathcal{L}$ is totally ordered by inclusion, so that $\cap \mathcal{L} \neq \square$. If d is a point of $\cap \mathcal{L}$, then $d \le x$ for all x of $X \backslash V$, since $d \in L[x]$ for all x of $X \backslash V$. This implies that

$X \setminus V \subset U[d]$. It is not difficult to verify that $X \setminus U[d] = \cup\{L[c] \mid c < d\}$. Then $X \setminus U[d] \subset V$ and is an open connected set containing p. Hence the proof is complete.

Suppose X is a D-space with a D-point p. We let $\mathfrak{U}$ be the collection of all U[b] for each point b of X. Then $\mathfrak{U}$ is a collection of closed subsets of X which is totally ordered by inclusion. If we define $M = \cap\mathfrak{U}$, then M is a non-empty closed subset of X, and is called the maximal level set of X.

8-8 Theorem. Let X be a D-space with D-point p. Then the following statements hold.

(1) X is irreducible between its D-point p and any point m of M.

(2) Each point $z \in X \setminus (M \cup \{p\})$ is a weak cut point of X. Furthermore, if M contains more than one point, then every point z, except D-point p, is a weak cut point of X.

(3) Each level set L_d, except $L_p = \{p\}$ and the maximal level set M, cuts X.

(4) If, furthermore, X is locally connected, then X is an arc.

Proof. (1) Let A be a subcontinuum in X which contains p and a point m of M. Since $m \in M$ we have $L[m] = X$. But L[m] is the minimal continuum which contains p and m so that $L[m] \subset A$ and hence $A = X$.

(2) From (1) we know that each point $z \in X \setminus (M \cup \{p\})$ is a weak cut point between p and m a point of M. In the case

that M contains more than one point, it is sufficient to prove that each point m of M is also a weak cut point. Let n be a point of M which is different from m. Then it can be easily verified that m is a weak cut point between p and n, since $L[n] = X$ is the minimal continuum which contains p and n.

(3) If we let $P = L[d]\setminus L_d$ and $Q = U[d]\setminus L_d$, then $X\setminus L_d = P \cup Q$ and both P and Q are nonempty sets since $M \subset Q$ and $p \in P$. Also $P^* \cap Q = \square$, since $P^* \cap Q \subset L[d] \cap Q = \square$. Similarly it is true that $P \cap Q^* = \square$. Thus $X\setminus L_d$ is a disconnected set, which completes the proof.

(4) We prove this part by steps.

(i) We show that for each $a \neq p$, there exists some point $a_0 \in L_a$ such that if V is an open neighborhood of a_0, then there exists a point $b \in V$ with $b < a_0$. Suppose not and let a be a point such that for each $\ell \in L_a$ there exists an open neighborhood V_ℓ of ℓ for which $\ell \leq b$ for all $b \in V_\ell$. Then $U[a] = (X\setminus L[a]) \cup (\bigcup_{\ell \in L_a} V_\ell)$ is both open and closed in X. Since $a \neq p$, $U[a]$ is a proper subset of X which is both open and closed in X, which is impossible because X is connected.

(ii) We prove that each level set L_a is a singleton set. Recall that $L_p = \{p\}$, since $L[p] = \{p\}$. So we assume that $a \neq p$ and L_a contains more than one point. By (i) there exists a point $a_0 \in L_a$ which satisfies the statement mentioned in (i). Since L_a contains more than one point, we let a_1 be a point in L_a which is different from a_0 and let V_{a_0} be a connected neighborhood of a_0 which is small enough that

$a_1 \notin V_{a_0}^*$. By (i) there is a point b in V_{a_0} such that $b < a_0$. Then $L[b] \cup V_{a_0}^*$ is a closed connected set containing p and a_0. Since p is a D-point, we have $L[a_0] \subset L[b] \cup V_{a_0}^*$, which implies that $a_1 \in L[b]$. But this is a contradiction since $b < a_0$ and $a_0 \approx a_1$.

(iii) From (3) and (ii) we know that each point $b \in X \backslash M \cup \{p\}$ is a cut point of X. From (ii) we know that M is a singleton set, hence let $M = \{m\}$. In order to prove that X is an arc, it is sufficient to prove that p and m are noncut points. So suppose p is a cut point, i.e., $X \backslash \{p\} = P \cup Q$, where P and Q are disjoint nonempty open sets, and $P^* \cap Q^* = \{p\}$. Since $P^* \cap Q^* = \{p\}$ and $P^* \cup Q^* = X$ is a connected set it is not difficult to prove that both P^* and Q^* are connected. Then both P^* and Q^* contain the D-point p, but neither of them contains the other as a subset, which contradicts the fact that p is a D-point. On the other hand we have $X \backslash \{m\} = \cup\{L[d] \mid d \in X \text{ and } d \neq m\}$ is a connected set, which implies that m is not a cut point. Hence the proof is complete.

8-9 Remark. (1) The converse of part (1) in Theorem 8-8 is not true. The space X in Remark 8-5 is irreducible between points $(-\frac{1}{\pi}, 1)$ and $(\frac{1}{\pi}, 1)$ but is not a D-space.

(2) In application 8-11, there are two D-spaces which are not locally connected.

8-10 Theorem. Let S be a D-space with a D-point p whose maximal level set M contains more than one point. If S satisfies

$S^2 = S \neq K$, then the D-point p is either a left or a right identity for S. Furthermore, if v is a midunit for S, then $v = p$.

Proof. It follows from Theorem 1-6 that there exists an idempotent e which is a $\mathcal{J}$-maximal element of S since S is compact and $S^2 = S$. It follows from Theorem 1-7 that e is not a weak cut point of S since, by assumption $S \neq K$. Thus e must be the only point of S which is not a weak cut point of, namely, $e = p$. Since by part (2) of Theorem 8-8 we know that every point, except the D-point p, is a weak cut point of S. Also it follows from Theorem 1-7 that e is the only $\mathcal{J}$-maximal element S. This implies that $SeS = S$, since e is an idempotent and hence $J(e) = SeS$.

On the other hand we know that each eS and Se is a subcontinuum and contains the D-point p, so that either $eS \subset Se$ or $Se \subset eS$. Without loss of generality we may assume $eS \subset Se$. Then we have

$$S = SeS \subset S(Se) = S^2e = Se.$$

In this case e is a right identity. Similarly if $Se \subset eS$ then e is a left identity.

Furthermore, if v is a midunit of S, then $v = p$, since v is also a $\mathcal{J}$-maximal idempotent in S.

8-11 Application. (1) Let the underlying space of S be defined as $S = \{(x,y) \mid y = \sin(\frac{1}{x}),\ 0 < x \leq \frac{1}{\pi}\} \cup \{(x,y) \mid x = 0,\ -1 \leq y \leq 1\}$, with the usual topology as seen in the following figure.

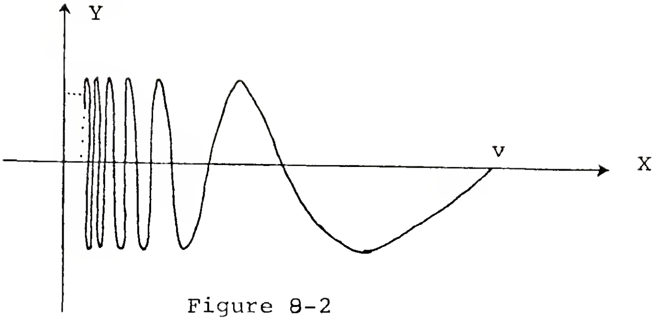

Figure 8-2

If S satisfies $S^2 = S \neq K$, then $v = (\frac{1}{\pi},0)$ is either a left or a right identity for S. Since S is a D-space with D-point $v = (\frac{1}{\pi},0)$ its corresponding maximal level set is $M = \{(x,y) | x = 0, -1 \leq y \leq 1\}$.

(2) Let $S = \{(e^{2\pi it}, e^{-t}) | t \in [0,\infty]\} \cup [C \times \{0\}]$ where C is a unit circle, with the usual topology as seen in the following figure.

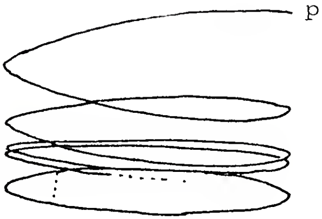

Figure 8-3

If S satisfies $S^2 = S \neq K$ then p is either a left or a right identity for S. Since S is a D-space with a D-point p, its corresponding maximal level set is $M = C \times \{0\}$.

We end this chapter with an example. This is an example of a continuum semigroup on a triod, probably the simplest continuum one can find which is not a D-space, which satisfies $S^2 = S \neq K$ and has a proper midunit.

8-12 Example. This example is constructed as follows. Let $T = \{v, a, b, 0\}$ with multiplication defined in the Table 4-1. Then T is a semigroup, with discrete topology on it, with a proper midunit v and satisfies $T^2 = T \neq K = \{0\}$. Let $I = [0,1]$ denote the closed real unit interval with the usual topology. Let $S_0 = T \times I$ with product topology and coordinatewise multiplication. Then S_0 is a semigroup with proper midunit $(v,1)$ and satisfies $S_0^2 = S_0$. If we let $S_1 = \{(v,0), (a,0), (b,0)\} \cup [\{0\} \times I]$, then S_1 is a closed ideal in S_0. Then the Rees quotient $S = S_0/S_1$ is a semigroup with proper midunit $(v,1)$ and zero satisfying $S^2 = S$, and is homeomorphic to a triod as in the following figure [21].

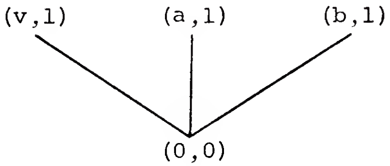

Figure 8-4

It is not difficult to see that this space is not a D-space, because none of the points in S can be a D-point.

CHAPTER IX

EXAMPLES OF SEMIGROUPS ADMITTING NO PROPER MIDUNIT

In this chapter we consider S to be a circle with a closed interval issuing from a point of the circle as seen in Figure 9-2. We prove that if S is a semigroup with midunit v and satisfies $S = SE \cup ES \neq K$, then v is either a left or a right identity for S and the circle is contained in K. The way of approaching this result is different from the one described in the last chapter. In working on this problem we found the interesting cohomological result given in Theorem 9-1. We also give two examples of semigroups which, due to the topological structure of their underlying spaces, do not admit the structure of a semigroup S satisfying $S^2 = S$ unless the multiplication on S is either left or right zero multiplication. We proceed with the main theorem of this chapter.

9-1 Theorem. Let S be a continuum semigroup with zero satisfying $S = ES \cup SE$. If $eS \cap fS$ and $Se \cap Sf$ are connected for each pair of idempotents e and f of S, then $H^1(S) = 0$.

Proof. Suppose the conclusion is not true; that is, there exists such a semigroup S such that $H^1(S) \neq 0$. Let h be a nonzero element of $H^1(S)$. Let F be a closed subset of E which is minimal relative to the condition $h|FS \cup SF \neq 0$. The existence of F is guaranteed by the assumption $S = ES \cup SE$ and the Reduction Theorem 1-2. It is claimed that F cannot be a singleton set. So we assume that $F = \{e\}$ where e is a idempotent such that $h|eS \cup Se \neq 0$. By the Mayer-Vietoris Sequence Theorem 1-1, we have the exact sequence,

$$\tilde{H}^0(eSe) \xrightarrow{\Delta} H^1(eS \cup Se) \xrightarrow{t^* \times t'^*} H^1(eS) \times H^1(Se) \xrightarrow{i^* - i'^*} H^1(eSe).$$

Since eSe is a subcontinuum semigroup with identity e and zero, one has $\tilde{H}^0(eSe) = 0$ and $H^1(eSe) = 0$ [19]. Therefore $t^* \times t'^*$ is an isomorphism from $H^1(eS \cup Se)$ onto $H^1(eS) \times H^1(Se)$. But eS and Se are subcontinuum semigroups with zero and left and right identity, respectively, and hence $H^1(eS) = 0$ and $H^1(Se) = 0$ [19]. Thus $H^1(eS \cup Se) = 0$, which contradicts the fact that $h|eS \cup Se \neq 0$. So one may write F as union of two closed proper subsets A and B, that is $F = A \cup B$. Then we have

$$\begin{aligned} FS \cup SF &= (A \cup B)S \cup S(A \cup B) \\ &= (AS \cup BS) \cup (SA \cup SB) \\ &= (AS \cup SA) \cup (BS \cup SB) \end{aligned}$$

and $(AS \cup SA) \cap (BS \cup SB) = (AS \cap BS) \cup (SA \cap BS) \cup (AS \cap SB) \cup (SA \cap SB)$.

Since $AS \cap BS = \cup\{eS \cap fS \mid e \in A \text{ and } f \in B\}$, $AS \cap BS$ is a union of connected sets with a common point zero, hence $AS \cap BS$ is a connected set. Similarly we can prove $SA \cap SB$ is a connected set. It is easy to see that $SA \cap BS = BSA$ and $SA \cap SB = ASB$ are connected sets. Therefore $(AS \cup SA) \cap (BS \cup SB)$ is a connected set in S.

Consider the following portion of the Mayer-Vietoris exact sequence.

$$\tilde{H}^0((AS \cup SA) \cap (BS \cup SB)) \xrightarrow{\Delta} H^1(FS \cap SF) \xrightarrow{t^* \times t'^*} H^1(AS \cup SA) \times H^1(BS \cup SB).$$

By the minimal condition on F we have $h|AS \cup SA = 0$ and $h|BS \cup SB = 0$, i.e., $t^* \times t'^*(h|FS \cup SF) = (h|AS \cup SA, h|BS \cup SB) = (0,0)$. On the other hand $\tilde{H}^0((AS \cup SA) \cap (BS \cup SB)) = 0$, since $(AS \cup SA) \cap (BS \cup SB)$ is a connected set. This implies that $t^* \times t'^*$ is injective so that $t^* \times t'^*(h|FS \cup SF) \neq (0,0)$, since $h|FS \cup SF \neq 0$. This is a contradiction and hence $H^1(S) = 0$. The proof is complete.

9-2 Remark. Without the condition that $eS \cap fS$ and $Se \cap Sf$ are connected for each pair of idempotents e and f of S, the conclusion of Theorem 9-1 is false by an example of Hudson [7]. This example is of a semigroup S = SE which is a continuum, has a zero and $H^1(S) \cong G$ for any coefficient group G. The underlying space of S is a circle with two closed intervals issuing from a common point of the circle as seen in the following figure.

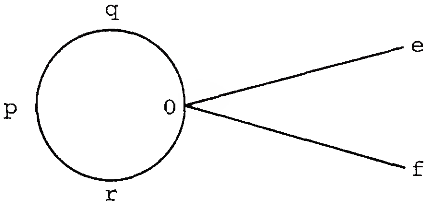

Figure 9-1

The points e and f are idempotents and 0 is the zero for S. Under the multiplication defined there, one has Se to be the arc from p to e containing q and 0, and Sf to be the arc from p to f containing r and 0. Also we have $Se \cap Sf = \{p,0\}$ which is a disconnected set.

If S is a continuum semigroup with zero satisfying S = ESE, then it is not difficult to prove that all left and right ideals of S are connected. In particular, $eS \cap fS$ and $Se \cap Sf$ are connected for each pair of idempotents e and f, since $eS \cap fS$ and $Se \cap Sf$ are right and left ideal of S, respectively. The following theorem due to Cohen and Koch [4] follows easily from Theorem 9-1.

9-3 Theorem. If S is a continuum semigroup with zero satisfying S = ESE, and if I is a closed ideal of S, then $H^1(I) = 0$. In particular, $H^1(S) = 0$.

The following example is an application of Theorem 9-1.

9-4 Example. Let S be a circle C with a closed interval I = [0,1] issuing from a point of the circle, as illustrated in the following figure.

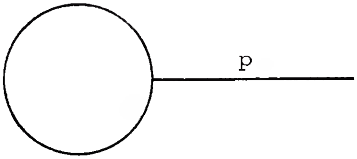

Figure 9-2

Suppose S = ESUSE. Then the following statements hold.

(1) If S = K, then S has either left zero or right zero multiplication.

(2) K contains the circle C.

(3) If $S \neq K$ and v is a midunit for S, then v is either a left or a right identity for S and is the endpoint of the protruding segment.

We prove this result as follows.

(1) If $S = K$ then since S admits neither a non-trivial topological factorization nor a topological group, there are only two possible multiplications on S; namely the left zero or right zero multiplication.

(2) In order to prove that K contains the circle C, we show first S does not admit the structure of a semigroup satisfying $S = ES \cup SE$ with zero. It is claimed that both $eS \cap fS$ and $Se \cap Sf$ are connected for each pair of idempotents e and f of S. Assume e and f are a pair of idempotents such that $eS \cap fS$ is a disconnected set and consider the following portion of the Mayer-Vietoris exact sequence.

$$\to \tilde{H}^0(eS) \times \tilde{H}^0(fS) \xrightarrow{i^*-i'^*} \tilde{H}^0(eS \cap fS) \xrightarrow{\Delta} H^1(eS \cup fS) \xrightarrow{t^* \times t'^*} H^1(eS) \times H^1(fS) \to$$

where $\tilde{H}^0(eS) = 0$, $\tilde{H}^0(fS) = 0$, $H^1(eS) = 0$ and $H^1(fS) = 0$, since both eS and fS are subcontinuum semigroups with zero and left identity [19]. Also we know that $\tilde{H}^0(eS \cap fS) \neq 0$, since by assumption $eS \cap fS$ is a disconnected set. It follows that $H^1(eS \cup fS) \neq 0$ since Δ is an isomorphism. It is not difficult to check that a subcontinuum B of S satisfies $H^1(B) \neq 0$ only if $C \subset B$. Hence $eS \cup fS$ is either the circle C or it is the circle C with part of the closed interval $[0,p]$ of I for some point p, as in Figure 9-2.

Suppose that neither $e \in fS$ nor $f \in eS$. If we consider $T = eS \cup fS$, then T is a subcontinnum semigroup satisfying $T = T^2$ and has e and f as two $\mathcal{R}$-maximal idempotents. It follows from Theorem 1-8 that neither e nor f can be a locally separating point of T. Since there is only one possible non-locally separating point in T (which is denoted by p in Figure 9-2), it follows that $e = f = p$. This contradicts the assumption that neither $e \in fS$ nor $f \in eS$. Therefore either $e \in fS$ or $f \in eS$. Without loss of generality we may assume $e \in fS$. Then we have that $eS \cap fS = fS$, which is again a contradiction since by assumption $eS \cap fS$ is disconnected. It follows that $eS \cap fS$ and $Se \cap Sf$ are connected for each pair of idempotents e and f of S. By Theorem 9-1 we have $H^1(S) = 0$. But we know that $H^1(S) \neq 0$ and hence S does not admit structure of a semigroup with zero satisfying $S = ES \cup SE$.

In order to prove that K contains the circle, we assume C is not contained in K. Then the underlying space of the Rees quotient S/K is homeomorphic to S again. Let E denote the set of idempotents in S/K, and let $\psi: S \rightarrow S/K$ be the natural homomorphism. Then $\psi(E) = E$ and it follows that $\psi(S) = \psi(E)\psi(S) \cup \psi(S)\psi(E)$ and has zero. By preceding argument we know this is impossible, thus C is contained in K.

(3) It follows from Theorem 1-8 that v has to be the endpoint of the closed interval I in Figure 9-2. Also since vSv meets K and $C \subset K$, one has $[S \backslash K] \subset vSv$, i.e., v is a two-sided identity for $S \backslash K$. Therefore, in order to prove that v is either a left or a right identity for S, it is

sufficient to prove that v is either a left or a right identity for K. Since K cannot be non-trivially topologically factored, there are only three possible multiplications on K; a group multiplication, left zero and right zero multiplications. In case K is a group with identity w, then $\{wv, vw\} \subset K$ since K is an ideal in S. Also we have $(wv)^2 = (wvw)v = w^2v = wv$ and $(vw)^2 = v(wvw) = vw^2 = vw$, i.e., wv and vw are both idempotents in K. But there is only one idempotent in K, hence $vw = wv = w$, i.e., v is a two-sided identity for K. Indeed, v is a two-sided identity for S when K is a group. In case K has left zero multiplication, then $kvk = k^2 = k$, and $(kv)k = kv$ for each k in K, which implies that v is a right identity for K. In case K has right zero multiplication, we can prove similarly that v is a left identity for S. Hence the proof is complete.

We conclude this chapter with two examples which, due to their topological structure, do not admit the structure of a topological semigroup S satisfying $S^2 = S$ unless S has left or right zero multiplication. The notion of midunit is used in the proof. The first example is due to McCharen [11]. We give his result here without proof.

9-5 Example. Let S be the space of tangent circles converging to a point e, as illustrated in the following figure.

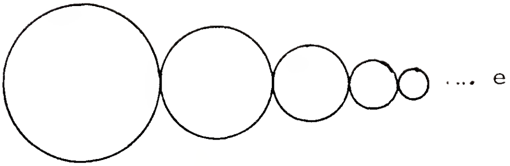

Figure 9-3

Then S does not admit the structure of a topological semigroup satisfying $S^2 = S$ unless S has of left or right zero multiplication.

Recall that a continuum X is I-semi-locally connected at point p, if for any open neighborhood U of p there exists an open set V such that $p \in V \subset U$ and $X \setminus V$ is connected. It is not difficult to prove that if X is a continuum and p is a marginal point of X, then X is I-semi-locally connected at p.

9-6 Example. Let S be defined by

$$S = \{(0,y)\,|\,-2 \le y \le 1\} \cup \{(x,-2)\,|\,0 \le x \le \tfrac{1}{\pi}\} \cup \{(\tfrac{1}{\pi},y)\,|\,-2 \le y \le 0\}$$

$\cup T$ where $T = \{(x,\sin\frac{1}{x})\,|\,0 < x \le \frac{1}{\pi}\}$, as illustrated in the following figure.

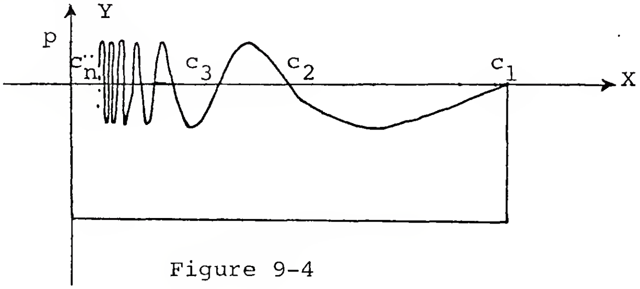

Figure 9-4

Then S does not admit a semigroup satisfying $S^2 = S$ unless S has either left or right zero multiplication.

We may assume $S \neq K$, since otherwise we are finished, since this space possesses no non-trivial topological factorization and, being nonhomogeneous, cannot be a group. By a sequence of observations we shall arrive at a contradiction. It is first claimed that S does not admit the structure of a topological semigroup S with midunit and zero also satisfying $S^2 = S$. If v is a midunit of S then from Proposition 3-1 we have that v is a $\mathcal{J}$-maximal idempotent in S. By Theorem 1-8 we know that a $\mathcal{J}$-maximal element cannot be a local separating point. It follows that v must be located at the point $p = (0,1)$, since p is the only non-local separating point in S. The sets vS and Sv are both continuum semigroups in S with zero, also with left identity and right identity, respectively. Hence $H^1(vS) = 0$, and $H^1(Sv) = 0$ [19]. Also it is not difficult to see that $H^1(S) \neq 0$, so that both vS and Sv are proper subcontinua of S and contain a common point $p = (0,1)$.

We try to prove that either $vS \subset Sv$ or $Sv \subset vS$. Let us consider $N = \{c_n \mid c_n = (\frac{1}{n\pi}, 0),\ n = 1,2,3,\ldots\}$ as illustrated in Figure 9-4. It is claimed that both vS and Sv could contain only finite many points of N. If vS contains infinitely many points of N, then it follows that vS contains the subset $\{(x, \sin\frac{1}{x}) \mid 0 < x < c\}$ of T for some c, since vS is a continuum containing p. Therefore vS is not I-semi-locally connected at p, otherwise it contradicts the fact that v is marginal in vS [11]. Hence both vS and Sv contain only finitely many points of N. Then it is easy to see that $vS \subset Sv$ or $Sv \subset vS$.

Without loss of generality we may assume $Sv \subset vS$, then $S = S^2 = (Sv)S \subset (vS)S = vS \subset S$; thus $S = vS$. Then S is a continuum semigroup with left identity and zero. Therefore one has $H^1(S) = H^1(vS) = 0$ [19]. This contradicts the fact that $H^1(S) \neq 0$.

Now we may assume K is not a singleton set. We consider the Rees quotient S/K, and let $\psi\colon S \to S/K$ be the natural homomorphism. Then it is evident that $\psi(S) = S/K$ is a continuum semigroup satisfying $(S/K)(S/K) = S/K$ and contains a midunit $\psi(v)$ and a zero. From the topological point of view we know that S/K is either homeomorphic to circle or to the original space as seen in Figure 9-4. But the case that S/K is homeomorphic to a circle cannot happen because every point on a circle is a local separating point. Hence S/K is homeomorphic to the original space S. From the preceding argument it follows that S/K does not admit a semigroup S/K with midunit and zero and satisfying $S^2 = S$.

Now we are able to prove that S does not admit a semigroup satisfying $S^2 = S$ unless S has either left or right zero multiplication. It follows from Theorem 1-8 that no $\mathcal{J}$-maximal element is a local separating point of S and hence p must be the unique $\mathcal{J}$-maximal element of S. By Theorem 1-6 we know p must be an idempotent. Therefore one has $S = SeS$. The conclusion follows from Proposition 7-1, Proposition 7-2 and preceding argument.

CHAPTER X

SEMIGROUPS WITH MIDUNIT ON THE TWO-CELL

In this chapter we investigate the structure of those semigroups with midunit and zero satisfying $S^2 = S$ on a topological two-cell. We let S be a two-cell with boundary B in a complex plane $\mathbb{C}$, M be the set of all midunits, J be the $\mathcal{J}$-class of midunits and H_v the maximal subgroup containing the midunit v. We prove that if $H^1(H_v) \neq 0$ for some midunit v, then $M = \{v\}$, $J = H_v = B$, and v is a two-sided identity for S. If $H^1(M) \neq 0$, then $J = M = B$, and all midunits are either left identities for S or all are right identities for S. It follows that $\mathrm{cd}(J) \leq 1$. We conclude this chapter with an example of a semigroup S with zero on a two-cell which satisfies $S^2 = S$ and contains a proper midunit.

Let $\mathbb{C}$ be a complex plane with the usual topology, and X be the unit two-cell in $\mathbb{C}$; that is, $X = \{z | z \in \mathbb{C} \text{ and } |z| \leq 1\}$. Let B be the boundary of X in $\mathbb{C}$; that is, $B = \{z | \ |z| = 1\}$. We proceed with a topological lemma.

10-1 Lemma. Let X be the unit two-cell with boundary B in a complex plane $\mathbb{C}$. If A is a closed subset of X such that $B \subset A$ and $H^1(A) = 0$, then $A = X$.

Proof. If $A \neq X$, then there exists an element $z_0 \in X \backslash A$.

Without loss of generality we may assume $z_0 = (0,0)$. If the function $\eta\colon A \to B$ is defined by $\eta(z) = \frac{z}{|z|}$, then η is continuous. It follows that B is a retract of A, since $\eta(b) = \frac{b}{|b|} = \frac{b}{1} = b$ for each b in B. If we let i be the inclusion map from B into A, then its induced homomorphism $i^*\colon H^1(A) \to H^1(B)$ is an epimorphism [13]. Therefore $H^1(A) \neq 0$, since $H^1(B) \neq 0$. But this is a contradiction, since by assumption $H^1(A) = 0$. The proof is complete.

It is well known that if A is a closed subset of the complex plane $\mathbb{C}$, then A cuts $\mathbb{C}$ if and only if $H^1(A) \neq 0$ [13].

10-2 Lemma. Let X be the unit two-cell with boundary B in a complex plane $\mathbb{C}$. If A is a closed subset of X such that $X\setminus A$ is a dense connected subset of X and $H^1(A) \neq 0$, then $B \subset A$.

Proof. Since by assumption $H^1(A) \neq 0$, we know that A cuts the complex plane $\mathbb{C}$. Then $\mathbb{C}\setminus A$ can be written as a union of a pair of nonempty separated sets P and Q; that is, $\mathbb{C}\setminus A = P \cup Q$. Without loss of generality we may assume that $X\setminus A \subset P$, since by assumption $X\setminus A$ is a connected set. Then it follows that $(X\setminus A)^* \subset P^*$, which is equivalent to $X \subset P^*$, since by assumption we know that $X\setminus A$ is a dense set in X. It is claimed that $P^* = P \cup A$. That $P \cup A$ is a subset of P^* is easy to see since $A \subset X \subset P^*$. On the other hand, since $P^* \cap Q = \square$ and $P \cup A \cup Q = \mathbb{C}$, we have that P^* is a subset of $P \cup A$. Therefore $P^* = P \cup A$, and $Q = \mathbb{C}\setminus P^*$. If we let $F[P^*]$ be the boundary of P^* in $\mathbb{C}$, then $F[P^*] = P^* \cap (\mathbb{C}\setminus P^*)^* = P^* \cap Q^* \subset A$, and $F[P^*] = P^*\setminus P^{*o} \subset P^*\setminus X^o \subset B \cup (P^*\setminus X)$. This implies that

$F[P^*] \subset A \cap [B \cup (P^*\backslash X)] = (A \cap B) \cup [A \cap (P^*\backslash A)] = A \cap B$, since $A \cap (P^*\backslash X) = \square$. It is true that $\mathbb{C}\backslash F[P^*] = P^{*o} \cup Q$, because $P^* \cup Q = \mathbb{C}$ and $F[P^*] = P^*\backslash P^{*o}$. On the other hand P^{*o} is a non-empty set, since X^o is a nonempty subset of P^{*o}. This implies that $F[P^*]$ cuts the complex plane $\mathbb{C}$, since $X\backslash F[P^*]$ can be written as a union of a pair of nonempty separated sets P^{*o} and Q. But no subset of B cuts $\mathbb{C}$ except B itself. This implies that $B \subset F[P^*]$ and therefore $B \subset A$ since we proved that $F[P^*] \subset A \cap B$.

10-3 Theorem. Let S be a two-cell with boundary B in the complex plane $\mathbb{C}$ with the usual topology. Suppose S is a semigroup with midunit and zero satisfying $S^2 = S$. Let M be the set of all midunits , J be the $\mathcal{J}$-class of the midunits, and H_v be the maximal subgroup of the midunit v. Then the following statements hold.

(1) If $H^1(H_v) \neq 0$ for some midunit v, then $J = H_v = B$, $M = \{v\}$, and v is a two-sided identity for S.

(2) If $H^1(M) \neq 0$, then $J = M = B$, and all midunits are either left identities for S or all are right identities for S.

(3) $\mathrm{cd}(J) \leq 1$.

Proof. (1) By Proposition 6-1 and Proposition 6-2 we know that $S\backslash H_v$ is a dense connected subset of X. It follows from Lemma 10-2 that $B \subset H_v$, since by assumption we know that $H^1(H_v) \neq 0$. Therefore $B \subset vSv$, since $H_v \subset vSv$. On the other hand we know that $H^1(vSv) = 0$, since vSv is a continuum subsemigroup with two-sided identity v and zero [19]. It follows from Lemma 10-1 that $S = vSv$. In order to prove $M = \{v\}$, let u be a midunit of S. Then we have $vuv = v^2 = v$ and also $vuv = u$, since v is a two-sided identity for S, therefore $u = v$ and $M = \{v\}$. Since v is a two-sided identity for S, we know that each point of H_v is a marginal point of S [6; p. 168]. This implies that $H_v \subset B$. But since we have proved that $B \subset H_v$, one has $H_v = B$. It follows easily from part (3) of Theorem 3-9 that $J = H_v$. The proof of (1) is complete.

(2) By Proposition 6-1 and Proposition 6-2 we know that $S\backslash M$ is a dense connected subset of X. It follows from Lemma 10-2 that $B \subset M$, since by assumption we know that $H^1(M) \neq 0$. Therefore $B \subset MSM$, since $M \subset MSM$. On the other hand we know that $H^1(MSM) = 0$, since MSM is a continuum subsemigroup with zero and M is a set of idempotents in S [4]. By Lemma 10-1 we know that $S = MSM$. It follows from Theorem 1-9 that $M \subset B$, since each midunit is a $\mathcal{J}$-maximal idempotent element of S. But since we have proved that $B \subset M$, one has $M = B$. By Proposition 2-4 we know that M is a rectangular band; i.e., $M \cong Mv \times vM$ with coordinatewise multiplication while Mv is a subsemigroup with left zero multiplication and vM is a subsemigroup with right zero multiplication, where v is a midunit. Since B is a unit circle which does not admit a non-trivial topological factorization, one has either $M \cong Mv$ or $M \cong vM$ [9]. In case M has left zero multiplication, it is claimed that all midunits are right identities for S. In order to prove this we let x be an element of S. Then there exists a pair

of midunits v and u such that $x = vxu$ since $S = MSM$. If w is a midunit, then $xw = vxuw = vx(uw) = vxu = x$, which implies that w is a right identity of S. Hence in this case all midunits are right identities for S. In case M has right zero multiplication we could prove analogously that each midunit is a left identity for S. It is true that if S is a continuum semigroup with zero satisfying $S = vS(Sv)$ for some idempotent v, then each point of H_v is marginal [11, p.6]. This implies that $H_v = \{v\}$ for each midunit v, since $B = M$ and each midunit is either a left or a right identity for S. It follows from Theorem 3-9 that $J = M$. The proof of part (2) is complete.

(3) If $H^1(H_v) \neq 0$ for some midunit v, then by (1) we have that $J = B$ so $cd(J) = 1$. We assume, therefore, that $H^1(H_v) = 0$ for each midunit v; that is, H_v is totally disconnected for each midunit v. From Corollary 5-5, it is sufficient to prove that $cd(M) \leq 1$. In order to prove $cd(M) \leq 1$, we let N be a closed subset of M such that $i^*: H^1(M) \to H^1(N)$ is not epimorphic, where i is the inclusion map from N into M. Therefore $H^1(N) \neq 0$. By Proposition 6-1 and Proposition 6-2 we know that $S\backslash N$ is a dense connected subset of X. It follows from Lemma 10-2 that $B \subset N$. Therefore $B \subset M$, since N is a subset of M. On the other hand we know that $H^1(MSM) = 0$, since MSM is a continuum subsemigroup with zero and M is a set of idempotents in S [4]. By Lemma 10-1 we know that $S = MSM$. It follows from Theorem 1-9 that $M \subset B$, since each midunit is a $\mathcal{J}$-maximal idempotent of S. Then we have

$M \subset B \subset N \subset M$, which implies that $M = N$. This contradicts the fact that i* is not epimorphic. This completes the proof.

10-4 Remark. (1) In [12], Mostert and Shields give a description of a semigroup on the two-cell where the boundary of the two-cell relative to the plane is a group, corresponding to the situation in part (1) of Theorem 10-3.

(2) In [10], Lester gives a description of a semigroup on the two-cell where the multiplication satisfies the following two conditions: (1) for x and y in the boundary B of the two-cell to the plane $xy = x$, and (2) there is a zero but no other idempotent in the interior of the two-cell. Then there exists an I-semigroup T in S such that $S = BT$. Also for e and f in B and s in T $(es)(ft) = e(st)$; and if $es = ft$ then $s = t$. This corresponds to a special case of part (2) of Theorem 10-3.

We end this chapter with an example of a semigroup S with zero and a proper midunit on a two-cell which satisfies $S^2 = S$.

10-5 Example. Let $\{a,b\}$ be a discrete space with left zero multiplication and $I = [0,1]$ denotes the real unit closed interval with usual multiplication. Then $\{a,b\} \times I$, with coordinatewise multiplication, is a compact semigroup. If we set $T_0 = (\{a\} \times I) \cup (\{b\} \times [0,\frac{1}{2}])$, then T_0 is a closed subsemigroup of $\{a,b\} \times I$. It is not difficult to see that $I_0 = \{(a,0),(b,0)\}$ is a closed ideal in T_0. If we let S_0 be the Rees quotient ${}^{T_0}/\{(a,0),(b,0)\}$, then S_0 is a semigroup on a closed unit interval with zero and with a right identity (a,1) at one endpoint which is not simultaneously a left identity for S_0, as seen in Figure 10-1. Let $S = S_0 \times S_0$ with multiplication be defined by $(x,y)(z,w) = (xz,wy)$. Then it can be verified that $S^2 = S$ with $v = ((a,1), (a,1))$ is a proper midunit. Since $S = S_0 \times S_0$, the underlying space of S is homomorphic to a two-cell.

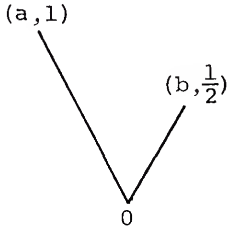

Figure 10-1

CHAPTER XI

MARGINALITY OF THE MIDUNITS IN A CONTINUUM SEMIGROUP

It is a well-known result that if S is a continuum semigroup with identity which is not a group, then each point of the maximal group of the identity is marginal in S [6;p.168]. In this chapter we investigate the marginality of the midunits in a continuum semigroup S satisfying $S^2 = S \neq K$. Recall that a point p in a space X is marginal if and only if for each open set U containing p, there exists an open set V such that $p \in V \subset U$ and $i^*: H^*(X) \to H^*(X \setminus V)$ is an isomorphism, where i is the inclusion map from $X \setminus V$ into X.

Without the condition $S^2 = S \neq K$, a midunit may not be a marginal point, as it is seen in the following example which was mentioned in Example 2-1.

11-1 Example. Let S be the closed unit interval [0,1] with the usual topology. If ab = minimum $\{a, b, \frac{1}{2}\}$, then S is a semigroup with $\frac{1}{2}$ as a midunit and 0 as zero. It is easy to see that $S^2 \neq S$ and $\frac{1}{2}$ is not a marginal point.

We proceed with a lemma.

11-2 Lemma. Let S be a compact semigroup with midunit v satisfying $S^2 = S \neq K$. Then the following two statements are equivalent.

(1) $H_v \subset (vSv)^o$

(2) $H_v \cap (Sv\backslash vSv)^* = \square$ and $H_v \cap (vS\backslash vSv)^* = \square$.

Proof. (1) $\Rightarrow$ (2) We know that $(vSv)^o \cap (Sv\backslash vSv) = \square$ since $(vSv)^o \subset vSv$, which implies that $(vSv)^o \cap (Sv\backslash vSv)^* = \square$ because $(vSv)^o$ is open. Therefore $H_v \cap (Sv\backslash vSv)^* = \square$, since by assumption we have $H_v \subset (vSv)^o$. Similarly we can prove that $H_v \cap (vS\backslash vSv)^* = \square$.

(2) $\Rightarrow$ (1) It is first claimed that $H_v \cap [(Sv\backslash vSv)^*(vS\backslash vSv)^*] = \square$. Otherwise we let x and y be a pair of elements such that $x \in (Sv\backslash vSv)^*$, $y \in (vS\backslash vSv)^*$ and $xy \in H_v$. If we let J be the $\mathcal{J}$-class of the midunit v, then it follows from Proposition 3-6 that $\{x,y\} \subset J$. It is true that $xv = x$ and $vy = y$, because $x \in (Sv\backslash vSv)^* \subset (Sv)^* = Sv$ and $y \in (vS\backslash vSv)^* \subset (vS)^* = vS$. Through use of Proposition 3-5 we know that $vyv \in H_v$. If z is the inverse of vyv in H_v, then we have $x = xv = xvyvz = xyvz = xyz \in H_v$. But this is a contradiction since $H_v \cap (Sv\backslash vSv)^* = \square$. Next we let $V = S\backslash[(Sv\backslash vSv)^* \cup (vS\backslash vSv)^* \cup (Sv\backslash vSv)^*(vS\backslash vSv)^*]$, and we want to prove that $H_v \subset V \subset (vSv)^o$. By assumption we have that $H_v \cap (Sv\backslash vSv)^* = \square$ and $H_v \cap (vS\backslash vSv)^* = \square$. Also we have proved that $H_v \cap [(Sv\backslash vSv)^*(vS\backslash vSv)^*] = \square$. It follows that $H_v \subset V$. In order to prove that $V \subset (vSv)^o$, it is sufficient to show that $V \subset vSv$, since V is open. Otherwise let z be a point such that $z \in V\backslash vSv$. Since $S^2 = S$ and v is a midunit, z can be written as $z = avb$ for some pair of elements a and b in S. It is claimed that $av \notin vSv$. So we assume that $av = vav$. Then $z = avb = vavb \in vS\backslash vSv$, which contradicts the fact that

$V \cap (vS\backslash vSv)^* = \square$. This proves that $av \notin vSv$. Similarly we can prove that $vb \in vS\backslash vSv$. Then

$z = avb = avvb \in (Sv\backslash vSv)(vS\backslash vSv) \subset (Sv\backslash vSv)^*(vS\backslash vSv)^*$.

But this contradicts the fact that $V \cap (Sv\backslash vSv)^*(vS\backslash vSv)^* = \square$. The proof is complete.

It has been proved that if A is a closed subspace of X and $p \in A^o$, then p is marginal in X iff p is marginal in A [11; p.3]. Hence the following theorem follows easily from Lemma 11-2.

11-3 Theorem. Let S be a continuum semigroup with midunit v satisfying $S^2 = S \neq K$. If $H_v \cap [(Sv \cup vS)\backslash vSv]^* = \square$, then each point of H_v is marginal in S.

Proof. It is evident that $H_v \cap [(Sv \cup vS)\backslash vSv]^* = \square$ is equivalent to $H_v \cap (Sv\backslash vSv)^* = \square$ and $H_v \cap (vS\backslash vSv)^* = \square$. By Lemma 11-2 we have $H_v \subset (vSv)^o$. Therefore each point of H_v is marginal in S, since each point of H_v is marginal in vSv.

CHAPTER XII

ACYCLICITY OF CONTINUUM SEMIGROUPS WITH MIDUNIT

In this chapter we are concerned with the acyclicity of a continuum semigroup with midunit. A continuum is called acyclic if it has the cohomology of a point space [18]. It has been known for some time that a continuum semigroup with left zero and left identity is acyclic [19]. The following example of a nonacyclic continuum semigroup with a zero and a midunit shows that, in the above, one may not replace the existence of a left identity with the existence of a midunit.

12-1 Example. Let $S = \{(x,y) \mid (x+1)^2 + y^2 = 1\} \cup \{(x,y) \mid 0 \leq x \leq 1, y = 0\}$ with the usual topology as seen in the Figure 9-2. Then S is a nonacyclic continuum, since $H^1(S) \neq 0$. We define multiplication on S by

$$pq = \begin{cases} (ab,0) & \text{if } p = (a,0),\ q = (b,0),\ a \text{ and } b \text{ are nonnegative, and } ab \text{ denotes the usual multiplication,} \\ (0,0) & \text{otherwise.} \end{cases}$$

Under this multiplication, S satisfies $S^2 \neq S$ and has a zero and a midunit (1,0).

To avoid this triviality we restrict ourselves to the study of the acyclic properties of those continuum semigroups satisfying $S^2 = S$ and having zero and a midunit. In working on this problem we found the following two interesting results which may be useful in attacking this problem from another point of view. We proceed with a lemma.

12-2 Lemma. Let S be a compact semigroup satisfying $S^2 = S$ $H^{n+1}(S) \neq 0$, and $\tilde{H}^n(I) = 0$ for any closed ideal I in S, where n is a nonnegative integer. If h is a nonzero element of $H^{n+1}(S)$, then there exists an idempotent e such that $h|SeS \neq 0$.

Proof. It has been proved that if S is a compact semigroup satisfying $S^2 = S$, then $S = SES$ [8]. Through use of the Reduction Theorem 1-2 it is possible to select a closed subset F of E, which is minimal relative to $h|SFS \neq 0$. Suppose F is not a singleton set, and let A and B be two nonempty proper closed subsets of F such that $F = A \cup B$. It is easy to see that $SFS = SAS \cup SBS$ and $SAS \cap SBS$ is a closed ideal in S. By the Mayer-Vietoris Sequence Theorem 1-1 we have the following exact sequence

$$\to \tilde{H}^n(SAS \cap SBS) \xrightarrow{\Delta} H^{n+1}(SFS) \xrightarrow{t^* \times t'^*} H^{n+1}(SAS) \times H^{n+1}(SBS) \to .$$

By the minimal condition on F, we have $(t^* \times t'^*)(h|SFS) = (h|SAS, h|SBS) = (0,0)$; thus $t^* \times t'^*$ is not monomorphic. But $SAS \cap SBS$ is a closed ideal of S, so that $\tilde{H}^n(SAS \cap SBS) = 0$ from our assumption, which implies that $t^* \times t'^*$ is monomorphic. This is a contradiction since we proved already that $t^* \times t'^*$ is not monomorophic. Thus F is a singleton set and the proof is complete.

12-3 Theorem. Let S be a continuum semigroup with zero. Then the following two statements hold.

(1) If S satisfies $S = ES \cup SE$ and h is a nonzero element of $H^1(S)$, then there exists an idempotent e such that $h|SeS \neq 0$.

(2) If S satisfies $S = ESE$ and h is a nonzero element of $H^2(S)$, then there exists an idempotent e such that $h|SeS \neq 0$.

Proof. (1) It is routine to check that each ideal I of S is connected since $S = ES \cup SE$, so that $\bar{H}^0(I) = 0$. The result then follows Lemma 12-2.

(2) It has been proved by Cohen and Koch [4] that if S is a continuum semigroup with zero satisfying $S = ESE$, and I is a closed ideal in S, then $H^1(I) = 0$. By Lemma 12-2, we know the proof is complete.

Recall that in Chapter VII we introduced a method of constructing new semigroups with midunit from given known semigroups. That is, if S is a semigroup and v is an idempotent in S, we define a multiplication "*" on S by $a*b = avb$. In particular, we proved in Proposition 7-1 and Proposition 7-2 that if S is a semigroup satisfying $SvS = S$ for an idempotent element v of S and S has a zero, then $(S;*)$ is a semigroup satisfying $S*S = S$, has v as a midunit, and has the same zero. For more details the reader is referred to Chapter VII.

From Theorem 12-3, we see that each example of a continuum semigroup with zero satisfying either $S = ES \cup SE$ with $H^1(S) \neq 0$ or $S = ESE$ with $H^2(S) \neq 0$ gives rise to a nonacyclic continuum semigroup with zero and midunit satisfying $S^2 = S$; namely $(SeS,*)$.

A. Hudson has given two examples of nonacyclic semigroups with zero, one satisfying $S = SE$ with $H^1(S) \neq 0$ and the other satisfying $S = ESE$ with $H^2(S) \neq 0$ [7]. The underlying space of the first example is a circle with two intervals issuing from a common point of the circle, as seen in Figure 9-1. The endpoints of those intervals are each midunits for S (use of Proposition 4-2 may be helpful in verifying this). We have then a nonacyclic continuum semigroup with zero and midunit satisfying $S = S^2$. The underlying space of the second example is a two-sphere with four intervals issuing from a common point from the sphere. A subsemigroup consisting of the two-sphere and two of the issuing intervals will have the endpoints of the intervals as midunits. This gives a second example of a nonacyclic continuum semigroup with zero and midunit satisfying $S^2 = S$.

It might be interesting to seek some sufficient conditions for continuum semigroup with zero and midunit under which S has to be acyclic. Toward this direction we introduce first a class of semigroups which we will call continuously factorable semigroups (see Definition 12-4). After a brief introduction of such class of semigroups, we prove that if S is a continuously factorable continuum semigroup, then all of its cohomology is concentrated in its minimal ideal. It follows that if S is a continuously factorable continuum semigroup with zero and midunit, then S is acyclic. We proceed with a definition.

12-4 Definition. Let $(S; m)$ be a topological semigroup with multiplication m. Then m and S are called continuously factorable if there exists a map $\eta: S \to S \times S$ such that

$m \circ \eta(s) = s$ for all elements s of S. We call η a continuous factorization of S.

The following are examples of continuously factorable semigroups.

12-5 Example. (1) A semigroup with left identity v is continuously factorable. For we could simply define $\eta(s) = (v,s)$ for all s of S. It is not difficult to check that η is a continuous factorization of S. A dual result holds for a semigroup with right identity. In particular, a clan is always continuously factorable.

(2) If S is a semigroup satisfying $S = E$, then S is continuously factorable. Naturally we define $\eta(s) = (s,s)$ for all elements s of S to obtain a continuous factorization of S.

(3) If S is a compact Clifford semigroup, then it is continuously factorable. Recall that a semigroup is called a Clifford semigroup if it is the union of groups. It has been proved that if S is a compact Clifford semigroup, then the map $\epsilon\colon S \to E$, which assigns to each s of S the identity of the group containing it, is continuous [6; p.44]. If we define $\eta(s) = (s,\epsilon(s))$ for all elements of S, then η is a continuous factorization of S.

We proceed with a theorem showing that all the cohomology of a continuum semigroup with midunit is concentrated in its minimal ideal if its multiplication is continuously factorable.

12-6 Theorem. Let $(S;m)$ be a continuously factorable continuum semigroup with midunit. If K is its minimal ideal, then the map $i: K \to S$ induces an isomorphism $i^*: H^*(S) \to H^*(K)$.

Proof. If $T = vSv$, where v is a midunit of S, then T is a clan with identity v. Since $T \cap K \neq \square$, there is an idempotent f in $T \cap K$. Let the two maps λ_v and λ_f from $S \times S$ into $S \times S$ be defined by $\lambda_v((x,y)) = (xv,y)$ and $\lambda_f((x,y)) = (xf,y)$ for all elements (x,y) of $S \times S$. On the other hand we consider a map $F: (S \times S) \times T \to S \times S$ defined by $F((x,y),t) = (xt,y)$ for all elements (x,y) in $S \times S$ and elements t in T. Then for all elements (x,y) of $S \times S$ we have $F((x,y),v) = (xv,y) = \lambda_v((x,y))$ and $F((x,y),f) = (xf,y) = \lambda_f((x,y))$. By applying Homotopy Theorem 1-4, we obtain $\lambda_f^* = \lambda_v^*: H^*(S \times S) \to H^*(S \times S)$.

Now let us consider the following diagram

$$\begin{tikzcd} S \arrow[r, "\eta"] & S \times S \arrow[r, shift left, "\lambda_v"] \arrow[r, shift right, "\lambda_f"'] & S \times S \arrow[r, "m"] & S \end{tikzcd}$$

, where m is the multiplication of S and η is a continuous factorization for m. For convenience we write η componentwise as $\eta(s) = (\eta_1(s),\eta_2(s))$. Then

$$\begin{aligned} m \circ \lambda_v \circ \eta(s) &= m \circ \lambda_v((\eta_1(s),\eta_2(s))) \\ &= m(\eta_1(s)v,\eta_2(s)) \\ &= \eta_1(s)\, v\, \eta_2(s) \\ &= \eta_1(s)\eta_2(s) \\ &= m \circ \eta(s) \\ &= s \end{aligned}$$

that is, $m \circ \lambda_v \circ \eta$ is the identity map I_S on S. On the other hand we have

$$\begin{aligned} (m \circ \lambda_f \circ \eta)^* &= \eta^* \circ \lambda_f^* \circ m^* \\ &= \eta^* \circ \lambda_v^* \circ m^* \end{aligned}$$

$$= (m \circ \lambda_V \circ \eta)^*$$
$$= (I_S)^* = I_{H^*(S)},$$

which implies that $(m \circ \lambda_f \circ \eta)^*: H^*(S) \to H^*(S)$ is an isomorphism.

Since $m \circ \lambda_f \circ \eta(s) = \eta_1(s) f \eta_2(s) \in SfS$ for all s of S, we may define $\ell: S \to SfS$ by $\ell(s) = m \circ \lambda_f \circ \eta(s) = \eta_1(s) f \eta_2(s)$ for all s of S. If i is the inclusion map of SfS into S, then it can be proved that $i \circ \ell(s) = m \circ \lambda_f \circ \eta(s)$ for all s of S. We then have $I_{H^*(S)} = (m \circ \lambda_f \circ \eta)^* = (i \circ \ell)^* = \ell^* \circ i^*$, which implies that $i^*: H^*(S) \to H^*(SfS)$ is a monomorphism, since $I_{H^*(S)}$ is an isomorphism. It is easy to see that K = SfS, since $f \in K$ and K is a minimal ideal of S. Since K is a retract of S [17], we also have that i* is also a epimorphism. The proof is complete.

The following corollary follows easily from Theorem 12-6.

12-7 Corollary. If S is a continuously factorable continuum semigroup with midunit and zero, then S is acyclic.

12-8 Remark. Corollary 12-7 shows that the examples of nonacyclic continuum semigroups with zero and midunit given earlier in this chapter are not continuously factorable.

REFERENCES

1. Ault, J. E., "Semigroups with midunits," Semigroup Forum, 6(1973), 346-351.

2. Clifford, A. H. and Preston, G. B., The Algebraic Theory of Semigroups, Vol. 1, Mathematical Surveys, 7, Amer. Math. Soc., 1961.

3. Cohen, H., "A cohomological definition of dimension for locally compact Hausdorff spaces," Duke Math. J. 21(1954), 209-224.

4. Cohen, H. and Koch, R. J., "Acyclic semigroups and multiplications on two-manifolds," Trans. Amer. Math. Soc. 118(1965), 420-427.

5. Faucett, W. M., Koch, R. J., and Numakura, K., "Complements of maximal ideals in compact semigroups," Duke Math. 22(1955), 655-662.

6. Hofmann, K. H. and Mostert, P. S., Elements of Compact Semigroups, Charles, E. Merrill, Inc. Columbus, Ohio, 1966.

7. Hudson, A. L., "Example of a nonacyclic continuum semigroup with zero and S = ESE," Proc. Amer. Math. Soc. 14(1963), 648-653.

8. Koch, R. J. and Wallace, A. D., "Maximal ideals in compact semigroups," Duke Math. J. 21(1954), 681-686.

9. Koch, R. J. and Wallace, A. D., "Admissibility of semigroup structures on continua," Trans. Amer. Math. Soc. 88(1958), 277-287.

10. Lester, A., "Some semigroups on the two-cell," Proc. Amer. Math. Soc. 10(1959), 648-655.

11. McCharen, J. D., Maximal Elements in Compact Semigroups, Dissertation, Louisiana State University, 1969.

12. Mostert, P. S., and Shields, A. L., "On the structure of semigroups on a compact manifold with boundary," Ann. of Math. 65(1957), 117-143.

13. Sigmon, K., Algebraic Topology Notes, Univ. of Florida, 1971.

14. Sigmon, K., "A strong homotopy axiom of Alexander cohomology," *Proc. Amer. Math. Soc.* *31*(1972), 271-275.

15. Stadtlander, D., *Topological Algebra Notes*, Univ. of Florida, 1972.

16. Wallace, A. D., "Inverses in Euclidean Mobs," *Math J.* Okayama Univ., *3*(1953), 23-28.

17. Wallace, A. D., "Retractions in semigroups," *Pacific J. Math.* 7(1957), 1513-1517.

18. Wallace, A. D., "A theorem on acyclicity," *Bull. Amer. Math. Soc.* *67*(1961), 123-124.

19. Wallace, A. D., "Acyclicity of compact connected semigroups," *Fund. Math.* *50*(1961), 99-105.

20. Wallace, A. D., *Project Mob*, Univ. of Florida, 1965.

21. Whyburn, G. T., *Analytic Topology*, Amer. Math. Soc., New York, 1952.

22. Yamada, Miyuki, "A note on middle unitary semigroups," *Kodai. Math. Sem. Rep.* *7*(1955), 49-52.

BIOGRAPHICAL SKETCH

Hung-tzaw Hu was born November 5, 1941, at Chungking, China. In June, 1967 he received the degree of Bachelor of Science with a major in Mathematics from the National Taiwan University. After his graduation he served as Mathematics Instructor during his military service in the Army of the Republic of China. Following his discharge from the army, September, 1968, he came to the United States and enrolled in the Graduate School of the University of Florida. In June, 1970, he received the degree of Master of Science with a major in Mathematics from the University of Florida. From June, 1970 until the present time he has pursued his work toward the degree of Doctor of Philosophy.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

K. N. Sigmon, Chairman
Assistant Professor of
Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

T. T. Bowman

T. T. Bowman
Assistant Professor of
Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

J. A. Draper

J. A. Draper
Assistant Professor of
Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

E. E. Shult

E. E. Shult
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

D. S. Stadtlander
Assistant Professor of
Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Mark C. Yang
Assistant Professor of
Statistics

This dissertation was submitted to the Department of Mathematics in the College of Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1974

Dean, Graduate School

UNIVERSITY OF FLORIDA
3 1262 08553 0177